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NEW DEVELOPMENTS ON THE p-WILLMORE ENERGY OF SURFACES

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Abstract. The p-Willmore energy $\mathcal{W}^p$, which extends the venerable Willmore energy by accommodating different powers of the mean curvature in its integrand, is a relevant geometric functional that bears both similarities and differences to its namesake. To elucidate this, some recent results in this area are surveyed. In particular, the first and second variations of $\mathcal{W}^p$ are given, and a flux formula is presented which reveals a connection between its critical points and the minimal surfaces. Finally, a model for the p-Willmore flow of graphs is presented, and this connection is visualized through computer implementation.

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1. Introduction

An important conformal invariant of immersed surfaces in $\mathbb{R}^3$ is known as the Willmore energy

$$\mathcal{W}^2(\Sigma) = \int_\Sigma H^2 \, \mathrm{d}S$$

which measures the failure of the surface Σ to be totally umbilic. Besides being the object of much mathematical study in recent decades (e.g. [3, 7, 10, 15, 16] and references), the Willmore energy has also demonstrated its utility in fields such as biology and biophysics, where adaptations like the Helfrich model for biomembranes have proven to be highly accurate models of observable behavior [8]. Due to this mathematical and physical relevance, it is reasonable to consider an extension of this functional which incorporates different powers of the mean curvature $H = (1/2)(\kappa_1 + \kappa_2)$ in the integrand. In particular, it is natural to wonder what similarities or differences are observed as the exponent of H changes. To that end, recall the p-Willmore energy introduced in [7]

$$\mathcal{W}^p(\Sigma) = \int_\Sigma H^p \, \mathrm{d}S, \qquad p \in \mathbb{Z}_{\geq 0}.$$

This functional, which reduces to the Willmore functional when $p = 2$, includes the important surface area and total mean curvature functionals as the cases $p = 0, 1$ respectively, while also extending the notion of Willmore energy to any positive integer power.

Remark 1. *Note that this definition can be extended to arbitrary real exponents by considering the unsigned quantity $|H|^p$ in the integrand.*

At first glance, it is clear that there are fundamental differences between the minimizers of, say, the surface area and the Willmore energy. For example, any minimal surface is Willmore, but the converse is certainly untrue. Indeed, the round sphere is not minimal, yet it is globally minimizing for the Willmore energy among closed surfaces of genus 0 [16]. It is therefore reasonable to ponder: does this dissimilarity extend to higher powers of p?. This survey paper will highlight that, in some sense, it does not. In particular, it follows from results here that when $p > 2$, any p-Willmore surface with $H \equiv 0$ on its boundary must be minimal. This suggests that the theory of p-Willmore surfaces when $p > 2$ is intimately related to the well-studied theory of minimal surfaces, which is a welcome connection. Since many interesting properties of minimal surfaces are already known, it is hopeful that more results of this nature will be discovered in the future.

2. The Variation of p-Willmore Energy

Since variational calculus provides useful analytical tools for studying the critical points of geometric functionals like $\mathcal{W}^p$, it is advantageous to carry out some computations which will generate necessary conditions that p-Willmore surfaces must satisfy. To this end, let Σ be a surface (with or without boundary) and consider a one-parameter family $\mathbf{r}$ of compactly supported immersions of Σ into an ambient space form $\mathbb{M}^3(k_0)$ of constant sectional curvature k_0. By reparametrizing if necessary, it may be assumed that $\mathbf{r} : \Sigma \times (-\varepsilon, \varepsilon) \to \mathbb{M}^3(k_0)$ is expressed normally to the surface as $\mathbf{r}(\mathbf{x}, t) = \mathbf{r}_0(\mathbf{x}) + t\, u(\mathbf{x})\mathbf{n}(\mathbf{x})$ where $u : \Sigma \to \mathbb{R}$ is a smooth function and $\mathbf{n} : \Sigma \to S^2$ is a unit normal field. If applicable, it may also be assumed that $\mathrm{d}\mathbf{r}(T\partial\Sigma) = \mathrm{d}\mathbf{r}_0(T\partial\Sigma)$ for each t, so that the variation fixes the tangent space at the boundary. With this in place, the first and second variations of the p-Willmore energy can be computed as in [7]. For the convenience of the reader, some computational details are recalled in the following Lemma.

Lemma 2. *Let $\varepsilon > 0$ and $\mathbf{r} : \Sigma \times (-\varepsilon, \varepsilon) \to \mathbb{M}^3(k_0)$ be a smooth family of compactly supported immersions of a surface Σ evolving with normal velocity $\delta\mathbf{r} = u\,\mathbf{n}$ where $\delta = (\mathrm{d}/\mathrm{d}t)\big|_{t=0}$ is the variational derivative operator. Then, there are the following evolution equations*

$$\delta g = -2u\,h, \qquad \delta g^{-1} = 2\,u\,\hat{h}, \qquad \delta(\mathrm{d}S) = -2Hu\;\mathrm{d}S,$$

$$\delta(2H) = \Delta u + 2u(2H^2 - K + 2k_0), \qquad \delta K = 2H\Delta u - \langle h, \operatorname{Hess} u\rangle + 2HKu,$$

$$\delta(\Delta f) = \Delta \dot{f} + 2u\langle h, \operatorname{Hess} f\rangle + 2u\langle \nabla H, \nabla f\rangle + 2\,h(\nabla u, \nabla f) - 2H\langle \nabla u, \nabla f\rangle,$$

where $f : M \times \mathbb{R} \to \mathbb{R}$ is smooth, $\mathrm{d}S$ is the volume form on Σ, Δu is the Laplacian of u with respect to the surface metric g, $\langle h, \operatorname{Hess} u\rangle$ is the scalar product between the second fundamental form $h : T\Sigma \times T\Sigma \to \mathbb{R}$ and the Hessian of u, and $\hat{h}$ is the (2,0)-tensor $g^{-1} \cdot g^{-1} \cdot h$ formed by twice contracting the shape operator with the metric inverse.

Proof: The proof is purely computational, and can be found e.g. in [7]. ■

Using Lemma 2, a lengthy but straightforward computation then yields the first two variations of the p-Willmore energy.

Proposition 3. *The first variation of $\mathcal{W}^p$ is given by*

$$\delta \int_\Sigma H^p\, dS = \int_\Sigma \Big(\frac{p}{2} H^{p-1}\Delta u + (2H^2 - K + 2k_0)pH^{p-1}u - 2H^{p+1}u\Big)\;\mathrm{d}S.$$

Moreover, the second variation of $\mathcal{W}^p$ at a critical immersion is

$$\begin{aligned}
\delta^2 \int_\Sigma H^p\;\mathrm{d}S &= \int_\Sigma \frac{p(p-1)}{4} H^{p-2}(\Delta u)^2\;\mathrm{d}S \\
&+ \int_\Sigma pH^{p-1}\big(h(\nabla u, \nabla u) + 2u\langle h, \operatorname{Hess} u\rangle + u\langle \nabla H, \nabla u\rangle - H|\nabla u|^2\big)\;\mathrm{d}S. \\
&+ \int_\Sigma \Big((2p^2 - 4p - 1)H^p - p(p-1)KH^{p-2} + 2p(p-1)k_0H^{p-2}\Big)u\Delta u\;\mathrm{d}S \\
&+ \int_\Sigma \Big(4p(p-1)H^{p+2} - 2(p-1)(2p+1)KH^p + p(p-1)K^2H^{p-2} \\
&\quad + 4(2p^2 - 2p - 1)k_0H^p - 4p(p-1)k_0KH^{p-2} + 4p(p-1)k_0^2H^{p-2}\Big)u^2\;\mathrm{d}S.
\end{aligned}$$

Remark 4. *When Σ is closed, we recover the Euler-Lagrange equation*

$$\frac{p}{2}\Delta H^{p-1} + p(2H^2 - K + 2k_0)H^{p-1}u - 2H^{p+1}u = 0$$

which characterizes p-Willmore surfaces as critical points of the first variation. This reduces to the well-known (see e.g. [3]) Willmore equation when $p = 2$ and the ambient space is $\mathbb{R}^3$

$$\Delta H + 2H(H^2 - K) = 0.$$

These expressions are previously known for the Willmore case $p = 2$ (c.f [15]), and can be recovered from general expressions in [14] when the ambient space is Euclidean. However, the expressions above have the advantage of being expressed entirely in terms of the basic geometric invariants that arise from the surface fundamental forms, making them quite accessible for computation. The utility of this aspect will be made clear in the next sections, where p-Willmore surfaces with boundary are discussed.

3. A BVP for p-Willmore Surfaces

An interesting class of problems arises when considering p-Willmore surfaces with prescribed boundary data. Several studies have previously been done on boundary-value problems for the surface area functional (e.g. [9, 13] and references) as well as for the Willmore functional (e.g. [5, 12] and references), so it is natural to consider how such problems behave for values of p different from 0 or 2. As it turns out, it can happen that the behavior of H on the boundary of a $\mathcal{W}^p$-critical surface Σ completely controls what happens on its interior. More precisely, we have proven the following result in [6].

Theorem 5. *When $p > 2$ is an integer, any p-Willmore surface $\Sigma \subset \mathbb{R}^3$ with boundary satisfying $H = 0$ on $\partial\Sigma$ must be minimal.*

This result is in obvious contrast to the case $p = 2$, where there are several known examples of non-minimal Willmore surfaces with zero boundary values e.g. [5]. To sketch its proof, we first mention a flux formula that can be computed from the first variation of $\mathcal{W}^p$ using an appropriate choice of test function.

Lemma 6. *Let $\mathbf{n}$ be a unit normal to the surface $\Sigma \subset \mathbb{R}^3$ and $\boldsymbol{\eta}$ the appropriately-oriented unit conormal to $\partial\Sigma$. Then (omitting volume elements), the following flux formula holds*

$$\frac{2(p-2)}{p}\int_\Sigma H^p = \int_{\partial\Sigma} \nabla_{\boldsymbol{\eta}}(H^{p-1})\langle \mathbf{r}, \mathbf{n}\rangle - H^{p-1}\big(\langle \nabla_{\boldsymbol{\eta}}\mathbf{n}, \mathbf{r}\rangle + (2/p)H\langle \mathbf{r}, \boldsymbol{\eta}\rangle\big).$$

Proof: See [6] for a derivation. ■

Theorem 5 follows quickly from this formula, as enforcing $H = 0$ on the boundary shows that the integral of some power of the mean curvature must be zero on the interior, and a short argument considering the positive and negative regions of H separately establishes the statement.

Remark 7. *Interestingly, it is also evident from Lemma 6 that there can be no closed p-Willmore surfaces immersed in* $\mathbb{R}^3$ *when* $p > 2$*. For if there were,* H^p *would have to vanish in the interior, and a similar argument yields that the surface must be minimal. However, there are no closed minimal surfaces in* $\mathbb{R}^3$*, a contradiction.*

4. Computational p-Willmore Flow of Graphs

Many noteworthy examples of surfaces with boundary immersed in $\mathbb{R}^3$ arise as the graph of a smooth function $u : \mathbb{R}^2 \to \mathbb{R}$. To illustrate some of the p-Willmore results that have been surveyed so far, it is useful to describe how such surfaces can be modeled using a computer. Taking inspiration from [1], this will be discussed in the context of the p-Willmore flow problem, where the goal is to find a surface parametrization

$$\mathbf{r}(\mathbf{x}, t) = (\mathbf{x}, u(\mathbf{x}, t))^T$$

such that $\dot{r} = -\delta\mathcal{W}^p$ for all t in some interval $(0, \tau]$. In this case, the basic geometry on Σ can be expressed as follows.

Lemma 8. *Let* $\Sigma = \{(\mathbf{x}, u(\mathbf{x}))\,;\, \mathbf{x} \in \Omega\}$ *be a surface given as the graph of a function* $u : \Omega \subset \mathbb{R}^2 \to \mathbb{R}$*, let* I *denote identity on* $\mathbb{R}^3$*, and let* $A := \sqrt{\det g}$ *denote the induced area element on* Σ*. Then, with the choice of unit normal vector* $\mathbf{N} = (1/A)(\nabla u, -1)^T$*, it follows that*

$$g_{ij} = \delta_{ij} + u_i u_j, \qquad A = \sqrt{1 + |\nabla u|^2}, \qquad g^{ij} = \delta^{ij} - \frac{u^i u^j}{A^2},$$
$$\Delta_\Sigma = \frac{1}{A}\nabla \cdot \left(A\left(I - \frac{\nabla u \otimes \nabla u}{A^2}\right)\nabla\right),$$
$$h_{ij} = \frac{u_{ij}}{A}, \qquad 2H = \nabla \cdot \left(\frac{\nabla u}{A}\right), \qquad K = \frac{\det(\operatorname{Hess} u)}{A^4}.$$

Proof: The proof is a straightforward calculation and can be found e.g. in [6]. ■

Using the above expressions and a trick employed in [4], it is possible (with moderate effort) to express the p-Willmore Euler-Lagrange equation

$$\frac{p}{2}\Delta_\Sigma H^{p-1} + p(2H^2 - K + 2k_0)H^{p-1}u - 2H^{p+1}u = 0$$

in divergence form. Indeed, the authors of [4] noticed that the Laplace operator on Σ can be expressed as

$$\Delta_\Sigma f = \nabla \cdot \left(\frac{1}{A}\left(I - \frac{\nabla u \otimes \nabla u}{A^2}\right)\nabla(Af)\right) - f\nabla \cdot \left(\frac{1}{A}\left(I - \frac{\nabla u \otimes \nabla u}{A^2}\right)\nabla A\right)$$

which simplifies the first variation considerably in the Willmore case. When applied to the expression in Proposition 3, analysis similar to that of [4] eventually yields the divergence form equation

$$\nabla \cdot \left(\frac{p}{2A} \left(I - \frac{\nabla u \otimes \nabla u}{A^2} \right) \nabla W + \frac{WH}{A^2} \nabla u \right) = 0$$

where $W = AH^{p-2}$ is the *p-weighted mean curvature* of Σ (c.f. [1]). This gives a necessary condition that must be satisfied on the interior of any graphical p-Willmore surface, and which can be used to define the p-Willmore flow. Moreover, the expression of this condition in divergence form has the worthwhile advantage of being relatively simple to implement on a computer. To that end, it is useful to reformulate the p-Willmore flow problem in the following weak form.

Proposition 9. *Let $\Sigma(t)$ be a family of surfaces evolving by p-Willmore flow with parametrization $\mathbf{r}(\mathbf{x}, t) = (\mathbf{x}, u(\mathbf{x}, t))^T$ for $\mathbf{x} \in \Omega$, and let $\mathbf{v} \otimes \mathbf{w} = \mathbf{v}\mathbf{w}^T$ denote the usual outer product on vector fields. Then, the following system of equations is satisfied for all $t \in (0, \tau]$ and for all $\varphi, \psi, \xi \in H^1(\Sigma(t))$*

$$\begin{aligned} 0 &= \int_\Omega \frac{\dot{u}}{A}\varphi - \left(\frac{p}{2A} \left(I - \frac{\nabla u \otimes \nabla u}{A^2} \right) \nabla W + \frac{WH}{A^2} \nabla u \right) \cdot \nabla \varphi, \\ 0 &= \int_\Omega 2H\psi + \left(\frac{\nabla u}{A} \right) \cdot \nabla \psi, \\ 0 &= \int_\Omega W\xi - AH^{p-1}\xi. \end{aligned}$$

Proof: This system follows from integration-by-parts and the discussion above. Details can be found in [6, Chapter 5]. ■

Turning this statement around, if functions u, H, W are found that satisfy the system in Proposition 9 for all t in some interval $(0, \tau]$, then u will be the profile function of a graph undergoing p-Willmore flow. Using the finite element multiphysics solver FEMuS [2], it is straightforward to implement the graphical p-Willmore flow in this way, and observe its numerical behavior. In particular, Figure 1 displays the 4-Willmore flow applied to a surface with fixed undulary boundary. Notice that when $H = 0$ is imposed on the boundary, apparent convergence to a minimal surface is observed.

Moreover, the system in Proposition 9 can be used to confirm the statement of Theorem 5. In particular, by removing dependence on the parameter t, steady-state simulations can be performed which are consistent with the statement of that theorem. Two examples are given in Figure 2, where the 4-Willmore energy is considered. Here it is confirmed that, as expected, graphical minimizers of this

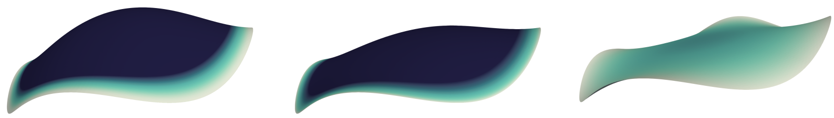

Figure 1. 4-Willmore evolution of a graph with undulary boundary and $H \equiv 0$ boundary data. Observe that the flow appears to converge to a minimal surface.

energy which satisfy $H = 0$ on a given boundary profile must in fact be minimal surfaces.

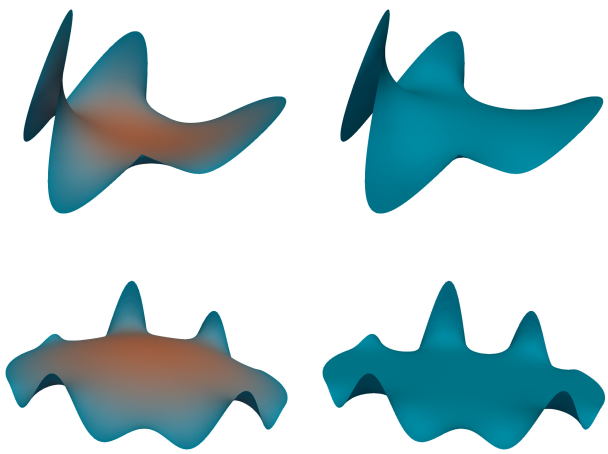

Figure 2. Solutions of the graphical 4-Willmore BVP with given boundary profile and $H \equiv 0$ boundary data. Initial data (left) and solutions (right). Note that the solutions are in fact minimal graphs.

Though convergence of the system in Proposition 9 has not yet been rigorously investigated, it has been shown that the graphical p-Willmore flow (with appropriate boundary conditions) does decrease the p-Willmore energy of a surface along its evolution. This is guaranteed by the following result from [6].

Theorem 10. *The graphical* p*-Willmore flow is energy-decreasing. That is, let* $\Sigma(t)$ *be a family of surfaces as above with profile function* $u(\mathbf{x}, t)$ *which obeys the* *p-Willmore flow equation*

$$\dot{u} + \frac{p}{2}A\nabla \cdot \left(\frac{1}{A}\left(I - \frac{\nabla u \otimes \nabla u}{A^2}\right)\nabla W\right) - A\nabla \cdot \left(WH\frac{\nabla u}{A^2}\right) = 0.$$

Then, if $u = g(\mathbf{x})$ is a fixed boundary curve and $H \equiv 0$ on $\partial\Sigma$, the p-Willmore energy satisfies

$$\int_{\Sigma(t)} \left(\frac{-\dot{u}}{A}\right)^2 + \frac{d}{dt}\int_{\Sigma(t)} H^p = 0.$$

Proof: See [6, Chapter 5]. ■

On the other hand, the p-Willmore flow is not always so well-behaved. In particular, it is not difficult to see that the p-Willmore energy is generally unbounded from below when p is odd, and this unboundedness is often observed in numerical simulation. As an example, Figure 3 illustrates what can happen when the p-Willmore flow is started from a surface of negative energy. This situation is easily constructed, since a surface with pointwise $H < 0$ can be recovered from a mean convex surface by simply making an alternate choice of normal vector. In this case, Theorem 10 ensures that the surface becomes more and more unstable along the p-Willmore flow, until the simulation eventually breaks. Consequently, in this case no convergence is observed at all.

Figure 3. 3-Willmore evolution of a graph with undulary boundary, beginning from a surface of negative 3-Willmore energy. Note that the flow becomes increasingly unstable and does not converge.

Though some things are known about the p-Willmore energy and its L^2-gradient flow, it is clear that many questions remain. Notably, almost nothing is known about the convergence or long-time behavior of the p-Willmore flow when $p > 2$, even in the somewhat limited case of surfaces expressible as a graph. Moreover, there are certainly numerous other relevant boundary conditions which can be imposed in conjunction with the p-Willmore equation, and future work should investigate the possibility of non-minimal critical surfaces in these cases as well. Finally, the notion of p-Willmore energy can be extended to surfaces of variable dimension and codimension as well as ambient spaces with nonconstant curvature, which may yield further interesting behavior that is unseen in the low-dimensional space forms considered so far.

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