Quaternionic Remeshing During Surface Evolution

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Abstract. This note discusses a recent method for remeshing along a numerical curvature flow. Based on ideas from quaternionic conformal geometry, this remeshing procedure ensures that a surface mesh does not become artificially degenerate as it evolves. Though developed primarily for use on immersed surfaces, preliminary results show promise for volumetric applications as well, leading to avenues for future work.

INTRODUCTION

Domain discretization is an inherent issue in the computer implementation of continuous phenomena, and it is well known that a good discretization is paramount to algorithmic accuracy and efficiency. This is especially true when considering numerical methods for geometric PDEs where the domain moves along with the solution. An important class of examples of this is afforded by curvature flows, which find use in several diverse areas of science and mathematics. Usually arising from the L^2 minimization of some curvature-based energy functional, curvature flows are used to model important natural phenomena such as diffusion [1, 2] and the behavior of biomembranes in solution [3, 4, 5]. Moreover, such flows have also found significant use in computer graphics applications due to their pleasing aesthetic character [6, 7].

Despite the widespread utility of curvature flows, there are several technical issues presented in both their theory and implementation that have yet to be fully overcome. For example, existence and regularity restrictions on solutions can manifest in flow problems that are ill-posed, and computational phenomena such as domain degeneration and the formation of singularities present known difficulties for numerical flow algorithms [2]. It is an important (and still not fully understood) challenge to ensure that an evolving surface remains well behaved along a numerical flow. Indeed, without some kind of regularization procedure in place, it is not uncommon for a nicely discretized initial domain to become degenerate before a particular curvature flow reaches an equilibrium configuration (e.g. [8] Figure 9). Moreover, even when an initial surface can flow completely to a minimum, poor mesh quality often results in an unacceptable loss of shape or texture (see Figure 2). Recent progress in this area has shown that ideas from quaternionic conformal geometry [9] can be useful for remeshing algorithms. It turns out that the additional algebraic structure of the quaternions allows for a nice expression of conformality which is computationally convenient to work

FIGURE 1: Quaternionic remeshing applied to a statue mesh (courtesy of the Aim@Shape repository) undergoing volume-constrained mean curvature flow. Images squeezed vertically by 20%. Note the immediate improvement over the initial triangulation (left image) as the surface evolves (others).

with. The remainder is dedicated to recent developments based on this idea, as well as applications to two-dimensional and three-dimensional domain remeshing.

REMESHING AN EVOLVING SURFACE

The remeshing technique employed here is formulated in [8] and based on the quaternionic work of [9] and the least-squares conformal mapping idea of [10]. In general, the method uses the algebraic structure of the quaternions to enforce an analogue of the Cauchy-Riemann equations from classical complex function theory in a least-squares sense. When applied to a surface undergoing curvature flow, this ensures that the evolving mesh does not degenerate as the surface moves.

More precisely, recall that any smooth function $f:\Sigma\to\mathbb{R}^3$ of the abstract surface Σ whose derivative is one-to-one may be thought of as an immersion which takes values in the imaginary part of the quaternions $\mathrm{Im}\,\mathbb{H}$. In this case, for any $\mathbf{x}\in\Sigma$ the tangent subspace $T_{\mathbf{x}}\Sigma$ can be made a complex vector space with complex structure $J:T_{\mathbf{x}}\Sigma\to T_{\mathbf{x}}\Sigma$, $J^2=-I$. Under this identification, it can be shown (see e.g. [9]) that f is conformal precisely when there exists a Gauss map $N:\Sigma\to S^2$ such that

$$*df = N\,df, \tag{1}$$

where $*\omega=\omega\circ J$ for all ω in the cotangent space $T^*\Sigma$. In other words, an immersion f is conformal if it induces a canonical complex structure J at each $\mathbf{x}$ which simply rotates the vectors in $T_{\mathbf{x}}\Sigma$ counterclockwise around $N(\mathbf{x})$. This gives an expression of the intrinsic notion of conformality in terms of the extrinsic notion of Gauss map which is highly advantageous for computation. In particular, in the numerical setting it is often more convenient to work with an immersed surface directly rather than constructing an artificial parametrization domain. So, there is high value in being able to manipulate intrinsic ideas such as conformality in an extrinsic way. Indeed, the relationship in (1) allows for the favorable adjustment of angles between the elements of a surface mesh, ensuring that they do not become degenerate as the surface changes. In the case of a curvature flow, this means that the initial surface cannot degrade artificially along its evolution. Hence, mesh quality will remain preserved provided the curvature flow does not reach a singularity.

To describe the quaternionic remeshing procedure more concretely, let g be the metric on Σ induced by the immersion f and $d\mu_g$ be its induced area element. The goal is to look for an immersion f which minimizes the conformal distortion,

$$\mathscr{C}\mathscr{D}(f)=\int_\Sigma |*df-N\,df|^2\,d\mu_g.$$

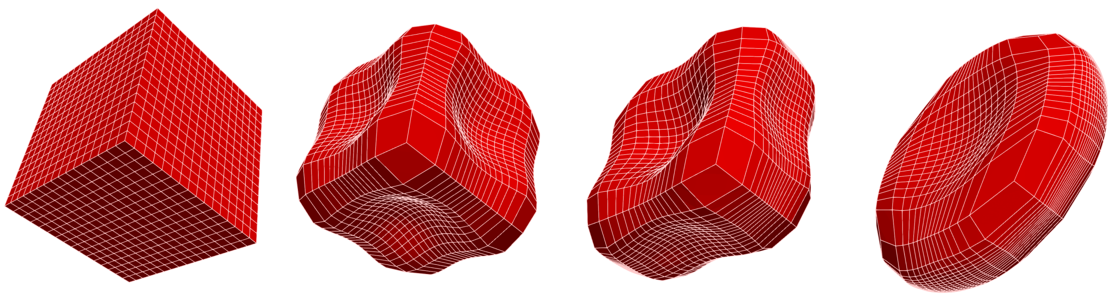
Obviously, the conformal distortion is simply the least-squares expression of (1), which can be handled with classical optimization techniques. By minimizing this quantity under an appropriate constraint, an evolving mesh will remain close-to-conformal with respect to a user-defined discretization of the reference domain. In practice, this discretization can be chosen "as flat as possible" (see [8]), so that the algorithm seeks a remeshing that is conformally-flat. This is equivalent to defining the configuration of zero distortion to be a grid of regular triangles or regular squares, depending on the element type of the surface. Though conformal flatness is usually impossible to attain exactly due to topological restrictions, this creates a regular and visually-pleasing discretization throughout the flow (c.f. Figure 1 and Figure 2).

Of course, an appropriate constraint is usually necessary during this procedure in order to preserve the desired surface geometry, since the minima of $\mathscr{C}\mathscr{D}$ are not guaranteed to leave unchanged the curvature properties which characterize the shape of the immersion. To describe one such constraint which is useful for this purpose, consider $x\in\Sigma$ and a curve $f(\mathbf{x}+t\mathbf{v})$ for some $\mathbf{v}\in T\Sigma$ and $0\leq|t|<\delta$. Then, it can be shown that the difference vector $f(\mathbf{x}+t\mathbf{v})-f(\mathbf{x})$ between $f(\mathbf{x})$ and a nearby point on the (smooth) immersed surface is second-order orthogonal to the "middle normal" vector $N(\mathbf{x}+(t/2)\mathbf{v})$. More precisely, it follows from Taylor expansion around $t=0$ that

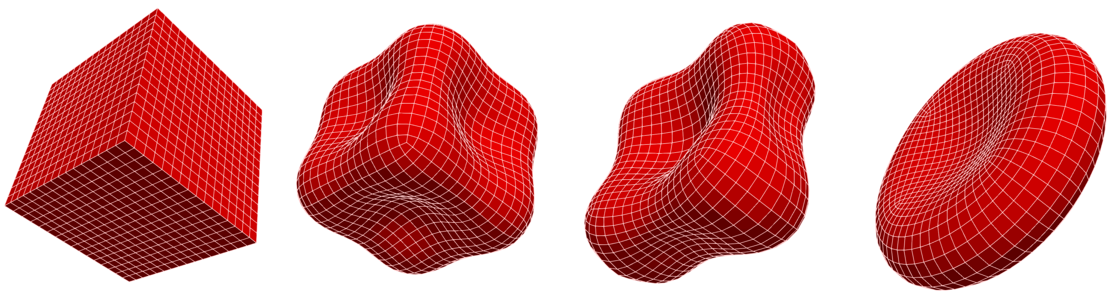
$$\begin{aligned} f(\mathbf{x}+t\mathbf{v}) &= f(\mathbf{x})+t\,df_{\mathbf{x}}(\mathbf{v})+\frac{t^2}{2}(\nabla df)_{\mathbf{x}}(\mathbf{v},\mathbf{v})+O(t^3),\\ N\left(\mathbf{x}+\frac{t}{2}\mathbf{v}\right) &= N(\mathbf{x})+\frac{t}{2}dN_{\mathbf{x}}(\mathbf{v})+O(t^2). \end{aligned}$$

Moreover, since $\langle N,df(\mathbf{v})\rangle=0$ for all $\mathbf{v}\in T\Sigma$, differentiation shows that $\langle dN(\mathbf{v}),df(\mathbf{v})\rangle+\langle N,(\nabla df)(\mathbf{v},\mathbf{v})\rangle=0$. Therefore, the inner product between the difference vector and the middle normal satisfies

$$\left\langle f(\mathbf{x}+t\mathbf{v})-f(\mathbf{x}),N\left(\mathbf{x}+\frac{t}{2}\mathbf{v}\right)\right\rangle = 0+O(t^3),$$



(a) Willmore flow without remeshing.



(b) Willmore flow with quaternionic remeshing.

FIGURE 2: Surface area and volume constrained Willmore flow applied to a cube. Here, our remeshing procedure (b) prevents degredation in mesh quality along the flow (a). Note that the minimizing immersion is a biconcave discoid characteristic of biomembranes which are critical for Helfrich-type energies [3, 5]

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and so this quantity vanishes to third order. Hence, for the purpose of remeshing it is reasonable to require that the difference vectors between the current and remeshed points on Σ remain orthogonal to the normal vectors in between. This relationship provides a reasonable and general constraint which can be readily discretized, and is implemented along with the minimization of $\mathscr{CD}$ as a Lagrange multiplier. Used appropriately, this keeps the remeshed immersion very close to the original, as any surface motion is almost completely tangential.

Collecting these ideas, the present remeshing procedure can be stated as follows. Given a fixed $\varepsilon > 0$ and an oriented surface immersion $f : \Sigma \to \mathbb{R}^3$ with outward unit normal field N, find a function $\nu : \Sigma \to \mathbb{R}^3$ and a Lagrange multiplier $\rho : \Sigma \to \mathbb{R}$, so that the new immersion $\hat{f} = f + \nu$ is a solution to

$$\min_{\nu} \left(\mathscr{CD}(f+\nu) + \frac{\varepsilon}{2} \int_{\Sigma} \rho^2 \, d\mu_g + \int_{\Sigma} \rho \, \langle \nu, N \rangle \, d\mu_g \right). \tag{2}$$

Remark 1 *Note that a penalty term involving ε has been introduced in order to aid in convergence of the numerical implementation. In particular, a parameter value of $\varepsilon = 1 \times 10^{-5}$ is seen to greatly improve convergence with negligible change in the resulting solution. Future work should investigate the influence of this parameter more carefully.*

To carry out the required minimization in (2), we apply standard techniques from the calculus of variations. Let $f : \Sigma \times \mathbb{R} \to \operatorname{Im}\mathbb{H}$, $f(\mathbf{x},t) = f_0(\mathbf{x}) + t\varphi(\mathbf{x})$ be a variation of the reference immersion f_0. Denote the standard basis for $T\Sigma$ as $\{\partial_i\}_{i=1}^2$ and the standard basis for $\operatorname{Im}\mathbb{H} \cong T\mathbb{R}^3$ as $\{\mathbf{e}_K\}_{K=1}^3$. Then, if $f_i^K = \langle df(\partial_i), \mathbf{e}_K \rangle_{\mathbb{R}}$, we can define the operators Q_1, Q_2 through (Einstein summation assumed)

$$\begin{aligned} \left(Q_1 f^K\right) \mathbf{e}_K &= *df(\partial_1) - N\,df(\partial_1) = \left(f_2^K - N f_1^K\right) \mathbf{e}_K, \\ \left(Q_2 f^K\right) \mathbf{e}_K &= *df(\partial_2) - N\,df(\partial_2) = -\left(f_1^K + N f_2^K\right) \mathbf{e}_K. \end{aligned}$$

Note that all products are considered as taking place between quaternions in $\mathbb{H}$. Minimization of $\mathscr{CD}$ (under fixed metric g) then yields the directional derivative at f in the direction φ,

$$\delta \mathscr{CD}(f)\varphi = \int_{\Sigma} \langle *df - N\,df, *d\varphi - N\,d\varphi \rangle_g \, d\mu_g := \int_{\Sigma} \langle Q(f), d\varphi \rangle_g \, d\mu_g,$$

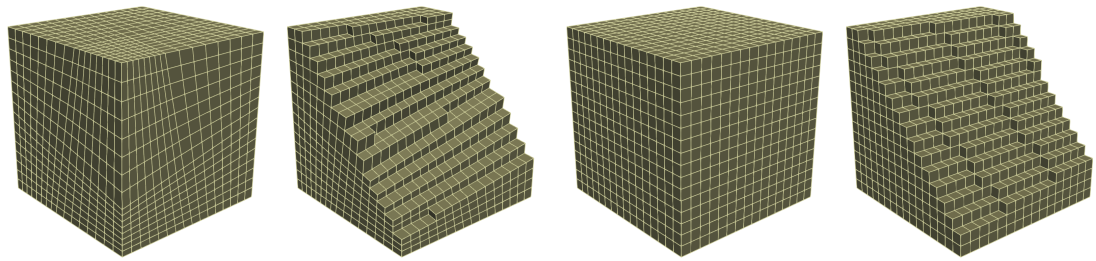

FIGURE 3: A heavily sheared cube (left two images) is regularized using the present method (right two images).

where the inner product $\langle Q(f), d\varphi\rangle_g$ is built according to

$$\langle Q(f), d\varphi\rangle_g = g^{ij} \left\langle \left(Q_i f^K\right) \mathbf{e}_K, \left(Q_j \varphi^L\right) \mathbf{e}_L \right\rangle_{\mathbb{R}} = -g^{ij}\,\mathrm{Re}\left(Q_i f^K \mathbf{e}_{K\times L} \overline{Q_j \varphi^L}\right),$$

and it was used that $\langle a,b\rangle_{\mathbb{R}} = \mathrm{Re}(a\bar{b})$, $\bar{\mathbf{e}}_K = -\mathbf{e}_K$, $\mathbf{e}_0 = 1$ and $\mathbf{e}_{-K} = -\mathbf{e}_K$. This allows for weak system of Euler-Lagrange equations for the minimization problem (2). More precisely, solving the quaternionic remeshing problem in weak form means finding a function $\nu : \Sigma \to \mathbb{R}^3$ and a multiplier $\rho : \Sigma \to \mathbb{R}$ which satisfy the system

$$\begin{aligned} 0 &= \int_\Sigma \langle Q(f+\nu), d\varphi\rangle_g \, d\mu_g + \int_\Sigma \rho \,\langle \varphi, N\rangle \, d\mu_g, \\ 0 &= \int_\Sigma \psi \,\langle \nu, N\rangle \, d\mu_g + \varepsilon \int_\Sigma \psi\rho \, d\mu_g, \end{aligned}$$

for all $\varphi \in H^1(\Sigma;\mathbb{R}^3)$ and for all $\psi \in H^1(\Sigma;\mathbb{R})$. This system, discretized as in [8], is used to generate the results seen presently. Clearly, the quaternionic remeshing procedure greatly improves surface quality along curvature flows. The specific examples in Figures 1 and 2 are generated using a modified implementation of the curvature flow methods due to Dziuk and Elliott [2] along with our procedure. As illustrated by Figure 2, remeshing during the Willmore flow leads to a much better discretization of the minimizing discoid. Moreover, Figure 1 shows that remeshing during the mean curvature flow can produce a triangulation which is even better than the original and persists throughout the evolution.

APPLICATION TO VOLUMETRIC REMESHING

The quaternionic remeshing procedure can also be extended to the case of three-dimensional mesh topologies with minor modification. Consider a hexahedral element with linearly independent edges $\{e_1, e_2, e_3\}$ at each vertex. Then, we can define f_{ij} to be the immersion of the plane spanned by e_i, e_j with Gauss map N_{ij}, and the total conformal distortion of each element to be the sum of the two-dimensional distortions in each unique (e_i, e_j)-plane. Integrated over the volume of the element, this gives a measure of total shear which can be minimized over a volumetric domain D to produce a more regular discretization. The (unconstrained) remeshing problem can then be stated as finding a function $\nu : D \to \mathbb{R}^3$ so that the immersion $\hat{f} = f + \nu$ satisfies

$$\min_\nu \sum_{i<j} \left(\int_D \left| * d\hat{f}_{ij} - N_{ij}\, d\hat{f}_{ij} \right|^2 dV \right).$$

Preliminary testing demonstrates that this procedure is meaningful, and successfully improves hexahedral meshes when the target volume is cubical. In particular, Figure 3 shows a cube with irregular elements which is made perfectly regular through this procedure. Future work involves developing a compatible constraint analogous to the surface case which is effective for shape preservation, as well as translating the quaternionic remeshing algorithm to more physically-involved settings such as simulations modeling fluid-structure interactions.

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