Willmore-Stable Minimal Surfaces

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Abstract. This note surveys recent progress on the Willmore and p-Willmore stability of minimal surfaces in three-dimensional space forms. In particular, the relationship of Willmore stability to classical stability is discussed for minimal surfaces in S^3, and it is shown that any minimal immersion into S^3 which is p-Willmore stable for some value of $p \geq 2$ is persistently p-Willmore stable.

INTRODUCTION

The Willmore energy (or bending energy), named after the mathematician Thomas Willmore, is a quantitative measure of how much an immersed surface deviates from a round sphere. Though originally discovered independently by Poisson and Germain at the beginning of the 19th century [1], its complete mathematical formalism was developed by Blachke, Thomsen [2], and later Willmore [3]. Now, the Willmore energy plays an important role in diverse areas such as digital image processing, elastic theory, theoretical liquid crystallography, and the theory of general relativity (see e.g. [4, 5, 6, 7] and the references therein).

Arguably the most important property of the Willmore energy is its invariance with respect to conformal transformations, which is useful in both the construction and exploration of Willmore surfaces. In some sense, this invariance is also the primary behavior that distinguishes the Willmore energy from other functionals depending on mean and Gaussian curvature. Often called curvature functionals or generalized Willmore energies, these objects include important examples such as the free energy of liquid crystals and the Hawking mass of subatomic particles. Indeed, let $\mathbf{r} : \Sigma^2 \to \mathbb{M}^3(c)$ be an immersion of the surface Σ into a 3-dimensional space form of constant sectional curvature $c \in \mathbb{R}$ with induced metric g and principal curvatures κ_1, κ_2. Then, the Gauss equation implies $(\kappa_1 - \kappa_2)^2 = 4\left(H^2 - K + c\right)$ where H, K are resp. the mean curvature and (intrinsic) Gauss curvature of the immersion. Under a conformal change of metric on $\mathbb{M}^3$ inducing the change $g \mapsto e^{2\varphi} g$ for some $\varphi : \Sigma \to \mathbb{R}$, a computation shows that $(\kappa_1 - \kappa_2)^2 \mapsto e^{-2\varphi}(\kappa_1 - \kappa_2)^2$. On the other hand, the induced change in the area element computes to $d\mu_g \mapsto e^{2\varphi} d\mu_g$, so that the integral of the density $\left(H^2 - K + c\right) d\mu_g$ becomes a conformal invariant. Moreover, when the surface Σ is closed, the usual definition of the Willmore energy

$$\mathscr{W}(\mathbf{r}) = \int_\Sigma \left(H^2 + c\right) d\mu_g,$$

is itself conformally invariant and does not depend on the intrinsic Gauss curvature of Σ, as the Gauss-Bonnet theorem assures this contribution to the integral is a topological constant. Real-world applications of $\mathscr{W}$ and other Willmore-type energies include the Helfrich energies introduced in biophysics for modeling closed lipid vesicles as elastic entities, cell membranes, and open lipid membranes [4, 5]. Note that all of these energies have in common a curvature functional whose integrand is a function of the mean curvature and Gaussian curvature, and which may also include other constants relevant to the physics. Some examples include rigidity constants, elasticity constants, spontaneous curvature, strain tensors, surface tension, and respectively the line tension of an edge in the case of an open membrane.

On the other hand, the above computations show that the Willmore functional is the only object of the form

$$\mathscr{W}^p(\mathbf{r}) = \int_\Sigma \left(H^2 + c\right)^{\frac{p}{2}} d\mu_g, \qquad p \geq 1,$$

which can be invariant under conformal changes of metric. The functional $\mathscr{W}^p$ is known as the p-Willmore energy, and is a natural Willmore-type generalization of the notion of p-energy in the classical analytic sense. In particular, recall that the p-Laplacian, or p-Laplace operator, is a quasilinear elliptic partial differential operator which arises during minimization of the p-energy functional—an L^p generalization of the standard Dirichlet energy which measures the integral of $|\nabla \mathbf{r}|^p$. The p-Willmore energy $\mathscr{W}^p$ has been considered by several authors, including ourselves, as a natural way to study the Willmore energy in the context of other interesting energy functionals which depend on curvature, such as the total mean curvature $\mathscr{W}^1$ and (with some abuse of notation) the area functional $\mathscr{W}^0$ [8, 9, 10].

In addition to its geometric interest, there is some motivation for $\mathscr{W}^p$ in mathematical biology, as there is an ongoing desire to include higher-order terms in the Helfrich model for the elastic energy of biomembranes which has been proposed on physical grounds [6, 7]. Note that for biological applications it is very natural to desire a measurement for surface energy that is not conformally invariant, as membranes in solution will have preferred shapes which depend on the metric itself and not just its conformal class. In fact, the resistance of biomembranes in equilibrium to small perturbations of their surroundings demonstrates the importance of stability in the study of energy functionals such as $\mathscr{W}$ and $\mathscr{W}^p$. It is frequently the case that the critical immersions for an energy functional will not be minimizers, but rather saddle points in the space of solutions. Though often interesting and beautiful on their own (see [11, 12]), these immersions are necessarily fragile. Even when the functional in question has strong physical relevance, its unstable critical points will not exist naturally for more than a few instants in time. Hence, it is advantageous to know beforehand which immersions will be stable for a given energy functional. This leads to a number of interesting questions which will be discussed in the sequel.

STABILITY OF MINIMAL WILLMORE IMMERSIONS

To study the stability properties of a functional, it is necessary to know information about its variational derivatives. More precisely, let $\mathbf{n}:\Sigma\to S^2$ be the outward unit normal to the compact surface $\Sigma\subset\mathbb{M}^3(c)$, and let $r:\Sigma\times(-\varepsilon,\varepsilon)\to\mathbb{M}^3(c)$ be a 1-parameter family of surface immersions such that $\mathbf{r}(x,t)=\mathbf{r}_0(x)+t\,u(x)\mathbf{n}(x)$ for some smooth function $u:\Sigma\to\mathbb{R}$. Then, the first variational derivative (or first variation) of the energy functional $\mathscr{F}$ at the immersion $\mathbf{r}$ and in the direction u is given by

$$\delta\mathscr{F}(\mathbf{r})u=\frac{d}{dt}\mathscr{F}(\mathbf{r}+t\,u\,\mathbf{n})\big|_{t=0},$$

and immersions at which this quantity vanishes (for all directions u) are said to be critical or stationary for $\mathscr{F}$. To understand which critical immersions are stable, it is necessary to have access to the second variation (defined analogously) which is a quadratic L^2-operator whose spectrum contains important information. For generalized Willmore functionals like those which depend on smooth functions of the mean and Gauss curvatures, the necessary variations can be easily recovered from the results in [10]. In particular, the first and second variations of $\mathscr{W}$ seen in the following proposition are a direct corollary of the more general results seen there.

Lemma 1 *Let $r:\Sigma\times(-\varepsilon,\varepsilon)\to\mathbb{M}^3(c)$ be a 1-parameter family of surface immersions as above. Then, the first and second variations of $\mathscr{W}$ can be expressed as follows (area elements omitted).*

$$\begin{aligned}\delta\mathscr{W}(\mathbf{r})u&=\int_\Sigma H\Delta u+2H\left(H^2-K+c\right)u,\\ \delta^2\mathscr{W}(\mathbf{r})(u,u)&=\int_\Sigma\frac{1}{2}(\Delta u)^2+\left(3c-H^2-2K\right)u\Delta u+2\left(H^2-K+c\right)\left(4H^2-K+2c\right)u^2\\ &\quad+\int_\Sigma 2H\left(2u\langle\operatorname{Hess}u,h\rangle+h(\nabla u,\nabla u)+u\langle\nabla H,\nabla u\rangle-H|\nabla u|^2\right).\end{aligned}$$

Remark 1 *As in [10], the second variation above has been computed at a $\mathscr{W}$-critical immersion of Σ.*

Recall that an important class of Willmore surfaces in $\mathbb{R}^3$ arise as the stereographic projection of minimal surfaces in the three-sphere S^3 [13, 14]. Although there are no closed minimal surfaces in Euclidean space, it is a well-known theorem of Lawson that any compact and orientable surface can be minimally embedded in S^3 [15]. Therefore, it follows from conformal invariance that any such surface can be realized as a Willmore surface in $\mathbb{R}^3$. This important result shows that there are embedded Willmore surfaces in $\mathbb{R}^3$ of arbitrary genus. However, the stability of these surfaces is not well understood even in this specific case. In fact, when $\Sigma\subset S^3(c)$ is minimally immersed, it has the dual interpretation as a both a minimal surface and a Willmore surface, and it is possible to compare the notions of stability that arise from each interpretation. We will see that there are key differences between the two, which relates to important questions in the theory of immersions.

First, note that at a closed surface $\Sigma\subset\mathbb{M}^3(c)$, the vanishing of the first variation in Proposition 1 implies the Euler-Lagrange equation

$$\Delta H+2H\left(H^2-K+c\right)=0,$$

which characterizes Willmore surfaces in $\mathbb{M}^3(c)$. Obviously, if $\mathbf{r}$ is a minimal immersion of Σ then this equation is automatically satisfied. Moreover, the second variation simplifies dramatically in this case, as we have

$$\delta^2\mathscr{W}(\mathbf{r})(u,u) = \frac{1}{2}\int_\Sigma (\Delta u + (4c-2K)u)\,(\Delta u + (2c-2K)u) = \frac{1}{2}\left\langle J\,(J+2c)\,u,u\right\rangle_{L^2},$$

where $J = -(\Delta+4c-2K)$ is the (L^2) self-adjoint Jacobi operator coming from the second variation of the area functional $\mathscr{A}$. More precisely, a computation shows that

$$\delta^2\mathscr{A}(\mathbf{r})(u,u) = -\int_\Sigma u\Delta u + (4c-2K)u^2 = \langle Ju,u\rangle_{L^2},$$

so that the minimal immersion $\mathbf{r} : \Sigma \to S^3$ is $\mathscr{A}$-stable when the operator J is positive semidefinite. This is precisely the classical notion of stability as a minimal surface, which has been studied extensively in the literature (e.g. [16, 17] and the papers contained within). Indeed, polarization of $\delta^2\mathscr{A}$ yields the usual Morse index form,

$$I_{\mathscr{A}}(\mathbf{r})(u,v) = \langle Ju,v\rangle_{L^2}, \qquad u,v\in C^\infty(\Sigma),$$

whose negative eigenvalues give the index of the immersion.

Remark 2 *It is common to see the Jacobi operator expressed as* $J = -\left(\Delta+2c+|h|^2\right)$ *where* $|h|^2$ *is the squared norm of the second fundamental form of* $\mathbf{r}$. *Since* $|h|^2 = \kappa_1^2+\kappa_2^2 = 4H^2-2K+2c$ *by the Gauss equation, these expressions are equivalent.*

On the other hand, it is clear from the discussion above that a surface Σ will be $\mathscr{W}$-stable when the operator $J\,(J+2)$ is positive semidefinite. This operator is also self-adjoint as a composition of self-adjoint operators, and therefore has a discrete spectrum of eigenvalues ascending to ∞. Hence there is also a Willmore index operator,

$$I_{\mathscr{W}}(\mathbf{r})(u,v) = \frac{1}{2}\left\langle J(J+2)u,v\right\rangle_{L^2}, \qquad u,v\in C^\infty(\Sigma),$$

whose spectrum determines the stability of minimal immersions as Willmore surfaces. Certainly, the notions of $\mathscr{A}$-stability and $\mathscr{W}$-stability are related but different, and the Jacobi operator J provides a useful device for their comparison. For example, it follows from this discussion that a minimal immersion $\mathbf{r}\subset S^3$ is $\mathscr{A}$-stable provided the spectrum of J is nonnegative, while $\mathbf{r}$ is $\mathscr{W}$-stable provided this spectrum does not meet the interval $(-2,0)$. It is known that all closed minimal submanifolds of S^3 are $\mathscr{A}$-unstable (see [18]), and that the vector fields in the kernel of $I_{\mathscr{W}}(\mathbf{r})$ arise only from isometries and conformal transformations (see [19]).

In contrast, there are several examples of $\mathscr{W}$-stable minimal surfaces in S^3 which are known to be $\mathscr{A}$-unstable [20, 21]. Consider the Clifford torus $T^2\subset S^3$,

$$T^2 = \{\frac{1}{\sqrt{2}}(e^{i\theta},e^{i\phi})\,|\,0\leq\theta,\phi<2\pi\}.$$

It is straightforward to verify that T^2 is both minimal and intrinsically flat, so that its Jacobi operator is given by $J = -(\Delta+4)$. Since J has negative eigenvalues, this surface cannot be $\mathscr{A}$-stable. However, it is easily checked that T^2 is $\mathscr{W}$-stable, as the smallest eigenvalue of $W = J(J+2)$ is zero (note that T^2 is the image of a harmonic immersion). Moreover, it is sometimes the case that stability for a particular immersion is guaranteed only among some class of variations with a certain geometric or physical meaning. As an example of this, consider the equator of S^3 which is minimally immersed with constant Gauss curvature $K=1$. This surface is naturally isometric to S^2, and its Laplacian Δ has eigenvalues $\lambda_k = k(k+1)$. It follows that the spectrum of J contains the negative value -2, so that the equator cannot be generically $\mathscr{A}$-stable. Geometrically, this reflects the fact that a sphere can decrease its surface area by shrinking at a constant rate. On the other hand, the only unstable eigenfunctions for J on S^2 correspond to normal variations with constant magnitude, which are obviously not volume-preserving. Therefore, the equatorial $S^2\subset S^3$ is $\mathscr{A}$-stable on the orthogonal complement of the constant functions, which includes all volume-preserving variations. Of course, it is also clear that the equator should be $\mathscr{W}$-stable as the global minimizer of the Willmore energy, which is confirmed by computing the eigenvalues of $J(J+2)$. The extent of the interplay between $\mathscr{A}$-stability and $\mathscr{W}$-stability is not yet well understood; it is an ongoing question to determine an exhaustive relationship between these notions.

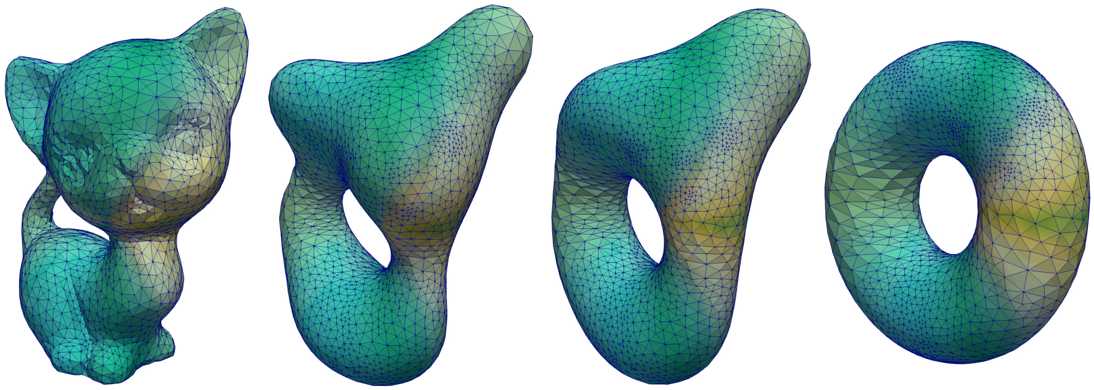

FIGURE 1. Area and volume preserving Willmore flow on a kitten mesh (courtesy of the Aim@Shape repository) produced with the algorithm in [22]. The stationary point is an appropriately constrained Clifford torus, which is conjectured to be the only stable minimizer among genus 1 surfaces.

STABILITY OF MINIMAL P-WILLMORE IMMERSIONS

Returning to the p-Willmore energy seen previously, it is natural to wonder what can be said about the stability of stationary solutions for this functional. In particular, it is evident that minimal surfaces are stationary for any value of p, but which of these surfaces (if any) are stable? Applying again the variational results from [10], there is the following lemma.

Lemma 2 *Let $p \geq 2$ and $c \in \{0,1\}$. At a minimal immersion $\mathbf{r} : \Sigma \to \mathbb{M}^3(c)$, it follows that the first variation $\delta \mathscr{W}^p(\mathbf{r}) \equiv 0$ and the second variation is*

$$\delta^2 \mathscr{W}^p(\mathbf{r})u = \left(\sqrt{c}\right)^{p-2} \int_\Sigma \frac{p}{4} (\Delta u)^2 + (p(2c-K)-c)\, u\Delta u + (2c-K)\,(p(2c-K)-2c)\, u^2.$$

In particular, when $\mathbf{r} : \Sigma \to S^3$, the second variation becomes

$$\delta^2 \mathscr{W}^p(\mathbf{r})u = \frac{p}{4} \int_\Sigma (\Delta u + (4-2K)\, u) \left(\Delta u + \left(4 - 2K - \tfrac{4}{p}\right) u\right) = \frac{p}{4} \left\langle J \left(J + \tfrac{4}{p}\right) u, u \right\rangle_{L^2},$$

where $J = -(\Delta + 4 - 2K)$ is the Jacobi operator on Σ.

Note that setting $p = 2$ recovers the corresponding expressions for the Willmore functional seen in Lemma 1. Moreover, it follows from Lemma 2 that an immersion $\mathbf{r} : \Sigma \to S^3$ is $\mathscr{W}^p$-stable provided its Jacobi operator J has no eigenvalues in the interval $\left(\frac{-4}{p}, 0\right)$. Since these intervals form a decreasing sequence under containment, we conclude that p-Willmore stability persists through increasing values of p. More precisely, there is the following result.

Proposition 3 *Any minimal immersion $\mathbf{r} : \Sigma \to S^3$ which is $\mathscr{W}^p$-stable for some value $p = p_0$ is also $\mathscr{W}^p$-stable for all $p > p_0$.*

This shows that minimal immersions into S^3 which are $\mathscr{W}$-stable are distinguished even among the larger family of energy functionals $\mathscr{W}^p$. This raises important questions worthy of future examination. In particular, precisely which minimial immersions are $\mathscr{W}$-stable? Surprisingly, of this remains unknown in general even among surfaces of genus zero [23]. There is strong evidence which supports that the round sphere (of any radius) is the only $\mathscr{W}$-stable minimizer of the Willmore energy among genus 0 surfaces, but this has not yet been proven. Additionally, it is conjectured that the Clifford torus is the only $\mathscr{W}$-stable minimizer among genus one, which also remains to be shown. Moreover, the connection between $\mathscr{W}$-stability and $\mathscr{W}^p$-stability highlighted in Proposition 3 raises interesting questions as well. Namely, for minimal immersions into S^3 which are $\mathscr{W}$-unstable, what is the smallest value of p which makes them $\mathscr{W}^p$-stable? The answers to these questions have the potential to dramatically increase our understanding of Willmore and generalized Willmore immersions, as well as to provide alternate and perhaps simpler proofs of the Willmore conjecture.

REFERENCES

1. S. Germain, *Recherches sur la théorie des surfaces élastiques* (Mme. Ve. Courcier, 1821).
2. W. Blaschke, K. Reidemeister, and G. Thomsen, *Vorlesungen über Differentialgeometrie und geometrische Grundlagen von Einsteins Relativitätstheorie* (J. Springer, 1929).
3. T. J. Willmore, "Note on embedded surfaces," An. Sti. Univ."Al. I. Cuza" Iasi Sect. I a Mat.(NS) B **11**, 493–496 (1965).
4. P. B. Canham, "The minimum energy of bending as a possible explanation of the biconcave shape of the human red blood cell," J. Theor. Biol. **26**, 61–81 (1970).
5. W. Helfrich, "Elastic properties of lipid bilayers: Theory and possible experiments," Z. Naturforsch., C, J. Biosci. **28**, 693–703 (1973).
6. K. Viswanathan and R. Parthasarathy, "A conformal field theory of extrinsic geometry of 2-d surfaces," Ann. Phys. **244**, 241–261 (1995).
7. J. C. C. Nitsche, "Boundary value problems for variational integrals involving surface curvatures," Quart. Appl. Math. **51**, 363–387 (1993).
8. Z. Guo, "Generalized Willmore functionals and related variational problems," Differential Geometry and its Applications **25**, 543–551 (2007).
9. A. Mondino, "Existence of integral m-varifolds minimizing $\int |A|^p$ and $\int |H|^p, p > m$ in Riemannian manifolds," Calc. Var. Partial Differential Equations **49**, 431–470 (2014).
10. A. Gruber, M. Toda, and H. Tran, "On the variation of curvature functionals in a space form with application to a generalized Willmore energy," Ann. Global Anal. Geom. **56**, 147–165 (2019).
11. U. Pinkall, "Hopf tori in s3," Invent. math **81**, 379–386 (1985).
12. E. Kuwert and J. Lorenz, "On the stability of the CMC Clifford tori as constrained Willmore surfaces," Ann. Global Anal. Geom. **44**, 23–42 (2013).
13. R. L. Bryant, "A duality theorem for Willmore surfaces," J. Diff. Geom. **20**, 23–53 (1984).
14. S. Brendle, "Minimal surfaces in S^3: a survey of recent results," Bull. Math. Sci. **3**, 133–171 (2013).
15. H. B. Lawson, "Complete minimal surfaces in S^3," Ann. Math. **92**, 335–374 (1970).
16. J. L. Barbosa, M. P. do Carmo, and J. Eschenburg, "Stability of hypersurfaces of constant mean curvature in Riemannian manifolds," Math. Z. **197**, 123–138 (1988).
17. M. do Carmo and K. Tenenblat, *Manfredo P. Do Carmo, Selected Papers* (Springer Berlin Heidelberg, 2012).
18. J. Simons, "Minimal varieties in Riemannian manifolds," Ann. Math. , 62–105 (1968).
19. J. L. Weiner, "On a problem of Chen, Willmore, et al," Indiana Univ. Math. J. **27**, 19–35 (1978).
20. F. Urbano, "Minimal surfaces with low index in the three-dimensional sphere," Proc. Amer. Math. Soc. , 989–992 (1990).
21. J. Choe and M. Soret, "First eigenvalue of symmetric minimal surfaces in S^3," Indiana Univ. Math. J. , 269–281 (2009).
22. A. Gruber and E. Aulisa, "Computational p-Willmore flow with conformal penalty," ACM Trans. Graph. **39**, 1–16 (2020).
23. L. Heller and F. Pedit, "Towards a constrained Willmore conjecture," in *Willmore Energy and Willmore Conjecture* (Chapman and Hall/CRC, 2017) pp. 119–138.