Codazzi tensors with parallel mean curvature

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Related work

Based on recent results from:

- 1) A. Gruber, "Parallel Codazzi Tensors with Submanifold Applications" (under review).

  Preprint available on arXiv or author's website at `www.myweb.ttu.edu/agruber`.

Inspired by ideas from:

- S. S. Chern, *Minimal Submanifolds in a Riemannian Manifold*. University of Kansas, Department of Mathematics, (1968).
- S. T. Yau, "Submanifolds with Constant Mean Curvature I & II". *American Journal of Mathematics*, (1974-1975).
- A. Derdzinski. "Some remarks on the local structure of Codazzi tensors". In *Global Differential Geometry and Global Analysis*. Springer, Berlin, Heidelberg (1981)

What is a Codazzi tensor?

Let $r : M^n \to \mathbb{M}^{n+p}(c)$ be a smooth immersion of the n-dimensional submanifold (M, g) into the $(n+p)$-dimensional space form $\mathbb{M}(c)$.

Adopt the index conventions,

$$1 \leq A, B, C \leq n+p, \qquad 1 \leq i, j, k \leq n, \qquad n+1 \leq \alpha, \beta, \gamma \leq n+p,$$

A *Codazzi tensor* on M is a symmetric tensor field $\varphi \in T^*M \otimes T^*M \otimes (TM)^\perp$ whose covariant derivative $\nabla\varphi$ is totally symmetric along M.

In a local ON frame $\{\mathbf{e}_A\} = \{\mathbf{e}_i\} \cup \{\mathbf{e}_\alpha\}$ where $\varphi = \sum \varphi^\alpha_{ij}\, \omega^i \otimes \omega^j \otimes \mathbf{e}_\alpha$, this is the componentwise condition

$$\varphi^\alpha_{ij,k} = \varphi^\alpha_{ik,j}.$$

Examples of Codazzi tensors

The second fundamental form h is the prototypical Codazzi tensor.

Recall that the Riemann curvature tensor of $\mathbb{M}(c)$ decomposes as

$$\tilde{R}^A_{BCD} = c\left(\delta^A_C \delta_{BD} - \delta^A_D \delta_{BC}\right).$$

Therefore, the classical Codazzi equation implies

$$h^\alpha_{ij,k} = h^\alpha_{ik,j} + \tilde{R}^\alpha_{ikj} = h^\alpha_{ik,j}.$$

General Codazzi tensors φ are modeled on h, and include:

- $\mathrm{Hess}_g\, f + c\, f\, g$ for any smooth $f : \mathbb{M}(c) \to \mathbb{R}$.
- Ric on manifolds with harmonic curvature, i.e. $\mathrm{div}_g(\mathrm{Riem}) = 0$.
- Any connection-parallel tensor field on a submanifold.

Rigidity theory: minimal submanifolds

Historically, study of the Laplacian Δh has been invaluable for determining the structure of *closed, minimal* Riemannian submanifolds.

Noticed originally by Bochner and later by Simons and Chern, computation of Δh leads to rigidity results for minimal immersions.

- (Simons 1968) Any closed minimal $M^n \subset S^{n+p}$ is either (1) totally geodesic, (2) has $|h|^2 = n/q$ where $q = 2 - 1/p$, or satisfies (3) $|h|^2 > n/q$ at some point.
- (Chern, do Carmo, Kobayashi 1970) The only surfaces satisfying (2) are the Clifford hypersurface and the Veronese surface.
- (Yau 1974) The hyperbolic space H^n cannot be minimally immersed in the Euclidean space $\mathbb{E}^{n+p}$, even locally.

Do such results hold for more general Codazzi tensors? In general, it is harder to say.

- (Derdzinski 1981) If M is complete, the eigenspace bundles of φ are generically integrable with totally umbilic leaves.
- (Derdzinski 1981) If M has dimension at least three and admits a nonparallel Codazzi tensor with constant trace and exactly two distinct eigenvalues, then M is locally isometric to a warped product.
- (Besse 2000) Suppose M admits a Codazzi tensor with constant trace and exactly two distinct eigenvalues with constant multiplicities greater than 1, then M is locally a Riemannian product.

Extending minimal submanifolds

Recently, a local-to-global rigidity result has been demonstrated under some additional assumptions.

Suppose M is not everywhere minimal and recall the *mean curvature vector*,

$$\mathbf{H} = \sum h_{ii}^{\alpha}\, \mathbf{e}_{\alpha} = \sum (\mathrm{tr}\, h^{\alpha})\, \mathbf{e}_{\alpha} = H\, \mathbf{e}_{n+1},$$

after a rotation in $(TM)^{\perp}$.

The submanifold M is said to have *parallel mean curvature* when

$$0 = \nabla^{\perp}\mathbf{H} = dH\, \mathbf{e}_{n+1} + H \sum \omega_{n+1}^{\alpha} \mathbf{e}_{\alpha},$$

so that H is constant and $\omega_{\alpha}^{n+1} = 0$ for all α.

This is a natural generalization of the CMC condition.

Main result

Consider the vector $\boldsymbol{\Phi} = \sum (\operatorname{tr} \varphi^\alpha) \, \mathbf{e}_\alpha = \phi \, \mathbf{e}_{n+1}$. We say φ has "parallel mean curvature" if $d\phi = 0$ and $\omega_\alpha^{n+1} = 0$ for all α.

Theorem (G.)

Let $M^n \subset \mathbb{M}^{n+p}(c)$ be a closed submanifold with nonnegative sectional curvature, and let φ be a Codazzi tensor on M. If the "mean curvature vector" $\boldsymbol{\Phi} = \phi \, \mathbf{e}_{n+1}$ is parallel in $(TM)^\perp$, then

$$|\nabla \varphi^{n+1}| = 0, \qquad \sum R^i_{jij} \left(\lambda_i^{n+1} - \lambda_j^{n+1} \right)^2 = 0.$$

Moreover, M is locally isometric to the product $M = M_1 \times M_2 \times \ldots \times M_l$ where l is the number of distinct eigenvalues of φ^{n+1} and TM_i is spanned by those eigenvectors which have eigenvalue λ_i^{n+1}.

If M is simply connected, then this statement is global.

Sketch of proof: part 1

Establish the commutation rule

$$\varphi^\alpha_{ij,kl} - \varphi^\alpha_{ij,lk} = \sum_{\beta,m} \left(\varphi^{\beta}_{ij} {}^{\perp}R^\alpha_{\beta lk} - \varphi^\alpha_{mj} R^m_{ilk} - \varphi^\alpha_{im} R^m_{jlk} \right).$$

It follows that

$$\begin{aligned}
\Delta \varphi^\alpha_{ij} &= \sum_k \varphi^\alpha_{ij,kk} = \sum_k \left(\varphi^\alpha_{ij,kk} - \varphi^\alpha_{ik,jk} \right) \\
&\quad + \sum_k \left(\varphi^\alpha_{ik,jk} - \varphi^\alpha_{ik,kj} \right) + \sum_k \left(\varphi^\alpha_{ik,kj} - \varphi^\alpha_{kk,ij} \right) + \sum_k \varphi^\alpha_{kk,ij} \\
&= \sum_k \left(\varphi^\alpha_{ij,kk} - \varphi^\alpha_{ik,jk} \right) + \sum_k \left(\varphi^\alpha_{ik,kj} - \varphi^\alpha_{kk,ij} \right) + \left(\operatorname{tr} \varphi^\alpha \right)_{,ij} \\
&\quad + \sum_{\beta,k,m} \left(\varphi^{\beta}_{ik} {}^{\perp}R^\alpha_{\beta kj} - \varphi^\alpha_{km} R^m_{ikj} - \varphi^\alpha_{im} R^m_{kkj} \right).
\end{aligned}$$

Skeptch of proof: part 2

Therefore, by the Codazzi equations

$$\Delta\varphi^{\alpha}_{ij} = (\operatorname{tr}\varphi^{\alpha})_{,ij} + \sum_{\beta,k,m}\left(\varphi^{\beta}_{ik}\,{}^{\perp}R^{\alpha}_{\beta kj} - \varphi^{\alpha}_{km}R^{m}_{ikj} - \varphi^{\alpha}_{im}R^{m}_{kkj}\right).$$

Let $\phi = \operatorname{tr}\varphi^{n+1}$. This yields the Simons-type formula

$$\frac{1}{2}\Delta|\varphi^{n+1}|^2 = |\nabla\varphi|^2 + \sum \lambda^{n+1}_i\,\phi_{,ii} + \frac{1}{2}\sum R^{i}_{jij}\left(\lambda^{n+1}_i - \lambda^{n+1}_j\right)^2,$$

where λ^{n+1}_i are the eigenvalues of φ^{n+1}. Since M is closed and φ has parallel mean curvature,

$$0 = \int_M |\nabla\varphi^{n+1}|^2 + \frac{1}{2}\int_M \sum_{i,j} R^{i}_{jij}\left(\lambda^{n+1}_i - \lambda^{n+1}_j\right)^2,$$

and the first conclusion follows.

Sketch of proof: part 3

This shows that each λ_i^{n+1} is constant, so the structure equations yield that $\omega_j^i = 0$ for $\lambda_i^{n+1} \neq \lambda_j^{n+1}$.

Lemma

(Derdzinski 1981) If $A = \sum A_j^i \omega^j \otimes e_i$ is a Codazzi tensor in (1,1)-form on the Riemannian manifold M and $\mathbf{u}, \mathbf{v}$ are eigenvectors of A with eigenvalue λ, then

$$A\left(\nabla_{\mathbf{v}}\mathbf{u}\right) = \lambda \nabla_{\mathbf{v}}\mathbf{u} + d\lambda(\mathbf{v})\mathbf{u} - \langle \mathbf{u}, \mathbf{v} \rangle \nabla \lambda.$$

Therefore, the eigenspaces corresponding to distinct λ_i^{n+1} are holonomy invariant and totally geodesic in M.

De Rham's decomposition theorem implies their maximal integral manifolds define a (local) product decomposition of M.

When M is simply connected, this decomposition is global. Q.E.D.

Consequences

- When $\varphi = h$, this was proved by Yau in 1974. Additionally, each factor lies in a totally umbilic submanifold of positive codimension in $\mathbb{M}(c)$.
- When the closed (M, g) has positive sectional curvature and φ has PMC, $\varphi^{n+1} = c\, g$ for some constant c and M is indecomposable.
- Particularly, if M is a closed hypersurface, then any Codazzi tensor with constant trace is a constant multiple of the metric.
- Let M be a closed, simply connected hypersurface with parallel Ricci tensor and $R^i_{jkl} \geq 0$. Then, M is globally a product with factors spanned by (exponentiated) EVs of Ric with distinct EW.
- Moreover, if the sectional curvature of M is strictly positive, then all eigenvalues of Ric are the same and M is both Einstein and indecomposable.

Questions for future work

- Yau proved in 1974 that closed *surfaces* $M \subset \mathbb{M}(c)$ with parallel mean curvature are either minimal or essentially codimension-one.

  In particular, they are either minimal in a totally umbilic hypersurface or lie in a totally geodesic 3-space. Is this true for submanifolds admitting more general Codazzi tensors?

- Can noncompact submanifolds or compact submanifolds with boundary be classified based on the behavior of general Codazzi tensors?

- What about other assumptions (such as flatness) which give some control on $(TM)^{\perp}$? What can be said in this case?

- Finally, what about the case of NO assumptions on $(TM)^{\perp}$? Can the method of studying $\Delta\varphi$ still be useful?

Thank you!