Willmore-stable minimal surfaces

Anthony Gruber

anthony.gruber@ttu.edu

Texas Tech University—Costa Rica

September 17, 2020

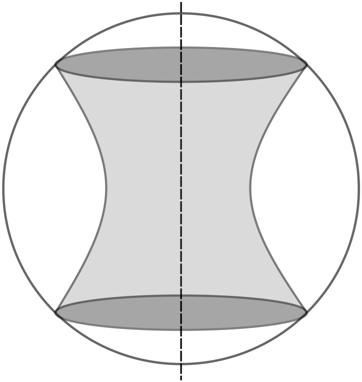
Source

Source material:

- 1) A. Gruber, M. Toda, H. Tran, "Willmore-stable minimal surfaces", *Proceedings of ICNAAM 2020.*
- 2) A. Gruber, M. Toda, H. Tran, "On the variation of curvature functionals in a space form with application to a generalized Willmore energy", *Annals of Global Analysis and Geometry*. July 2019, Volume 56, Issue 1, pp 147-165, `https://doi.org/10.1007/s10455-019-09661-0`.
- 3) R. Kusner, CIRM lecture, "Willmore stability and conformal rigidity of minimal surfaces in S^n", CIRM 2019, DOI: 10.24350/CIRM.V.19532803.

Author's preprints available at personal website: `http://myweb.ttu.edu/agruber`.

Outline



Origin of interest: Bending energy

Let $\mathbf{r} : M \to \mathbb{M}^3(c)$ be a smooth immersion of the surface M with induced metric g, and let κ_1, κ_2 denote the principal curvatures of this immersion.

The study of curvature functionals grew primarily from a model for **bending energy** proposed by Sophie Germain in 1821,

$$\overline{\mathcal{B}}(\mathbf{r}) = \int_M S(\kappa_1, \kappa_2)\, d\mu_g,$$

where S is a symmetric polynomial. By Newton's theorem, this is equivalent to the functional

$$\mathcal{B}(\mathbf{r}) = \int_M F(H, K)\, d\mu_g,$$

where F is polynomial in the mean and Gauss curvatures,

$$H = \frac{\kappa_1 + \kappa_2}{2}, \qquad K = \kappa_1 \kappa_2.$$

The Willmore energy

The simplest quadratic bending energy model of the form $\mathcal{B}$ is the (conformal) **Willmore energy**,

$$\overline{\mathcal{W}}(\mathbf{r}) = \frac{1}{4}\int_M (\kappa_1 - \kappa_2)^2 \, d\mu_g = \int_M \left(H^2 - K + c\right) \, d\mu_g.$$

When M is closed, the Gauss-Bonnet theorem implies that this has identical extrema to the functional

$$\mathcal{W}^2(\mathbf{r}) = \int_M \left(H^2 + c\right) \, d\mu_g.$$

(We will refer to $\overline{\mathcal{W}}, \mathcal{W}^2$ as the conformal Willmore energy and the Willmore energy, respectively.)

Critical points of $\mathcal{W}^2$ are called **Willmore surfaces** (or Willmore immersions), and arise frequently in biology and physics.

Examples of Willmore-type energies

- Helfrich-Canham energy,

$$E_{HC}(\mathbf{r}) := \int_M k_c(2H + c_0)^2 + \overline{k}K \, d\mu_g.$$

- Bulk free energy density,

$$\sigma_F(\mathbf{r}) = \int_M 2k(2H^2 - K) \, d\mu_g.$$

- Hawking mass,

$$m(\mathbf{r}) = \sqrt{\frac{\text{Area}\, M}{16\pi}} \left(1 - \frac{1}{16\pi} \int_M H^2 \, d\mu_g \right).$$

When M is closed, all share stationary surfaces with $\mathcal{W}^2$!

Area and volume preserving 2-Willmore flow of a cow

Basic questions

Given a $\mathcal{B}$-energy functional, it is natural to wonder about the following:

- What sort of immersions are critical for $\mathcal{B}$?
- Among these critical immersions, which are actually minimizing for $\mathcal{B}$.

Analytically, this involves some study of the variational derivatives.

Let $\mathbf{n} : M \to S^2$ be normal, $u : M \to \mathbb{R}$ and

$$\delta\mathcal{B}(\mathbf{r})u = \frac{d}{dt}\mathcal{B}\left(\mathbf{r} + t\, u\, \mathbf{n}\right)\Big|_{t=0}.$$

We will say a surface immersion is $\mathcal{B}$-**critical** provided $\delta\mathcal{B}(\mathbf{r})u = 0$ for all $u \in C^\infty(M)$.

Moreover, an immersion is $\mathcal{B}$-**stable** provided $\delta^2\mathcal{B}(\mathbf{r})(u, u) \geq 0$ for all $u \in C^\infty(M)$.

General second variation

Theorem: G., Toda, Tran

At a critical immersion of M, the second variation of $\mathcal{B}$ is given by

$$\begin{aligned}
\delta^2 \int_M F(H,K)\,dS &= \int_M \left(\frac{1}{4}F_{HH} + 2HF_{HK} + 4H^2F_{KK} + F_K\right)(\Delta u)^2\,dS \\
&+ \int_M F_{KK}\langle h, \text{Hess}\,u\rangle^2\,dS - \int_M (F_{HK} + 4HF_{KK})\Delta u\langle h, \text{Hess}\,u\rangle\,dS \\
&+ \int_M F_K\left(u\langle\nabla K, \nabla u\rangle - 3u\langle h^2, \text{Hess}\,u\rangle - 2\,h^2(\nabla u, \nabla u) - |\text{Hess}\,u|^2\right)dS \\
&+ \int_M \Big((2H^2 - K + 2k_0)F_{HH} + 2H(4H^2 - K + 4k_0)F_{HK} + 8H^2KF_{KK} \\
&\quad - 2HF_H + (3k_0 - K)F_K - F\Big)u\Delta u\,dS \\
&+ \int_M \Big((2H^2 - K + 2k_0)^2F_{HH} + 4HK(2H^2 - K + 2k_0)F_{HK} + 4H^2K^2F_{KK} \\
&\quad - 2K(K - 2k_0)F_K - 2HKF_H + 2(K - 2k_0)F\Big)u^2\,dS \\
&+ \int_M (2F_H + 6HF_K - 2(2H^2 - K + 2k_0)F_{HK} - 4HKF_{KK})u\langle h, \text{Hess}\,u\rangle\,dS \\
&+ \int_M (F_H + 4HF_K)h(\nabla u, \nabla u)\,dS + \int_M F_H\,u\langle\nabla H, \nabla u\rangle\,dS \\
&- \int_M (2(K - k_0)F_K + HF_H)|\nabla u|^2\,dS,
\end{aligned}$$

where the subscripts F_{HH}, F_{HK}, F_{KK} denote the second partial derivatives of F in the appropriate variables.

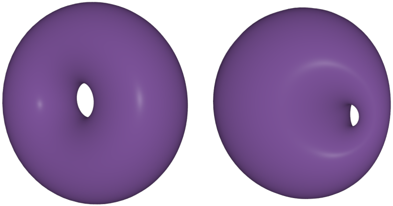
Connection: minimal and Willmore surfaces

$\mathcal{W}$-critical immersions (i.e. Willmore surfaces) satisfy

$$\Delta H + 2H(H^2 - K + c) = 0,$$

so any minimal surface is also Willmore. Moreover,

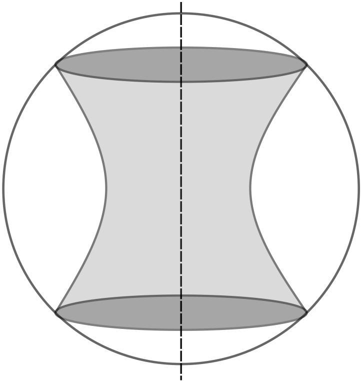
- The Willmore energy is invariant under conformal transformations of $\mathbb{R}^3 \cup \{\infty\}$.
- Stereographic projections of minimal surfaces in S^3 are Willmore surfaces in $\mathbb{R}^3$.



Since any closed surface can be minimally immersed in S^3 (Lawson 1970), this gives closed Willmore surfaces in $\mathbb{R}^3$ of arbitrary genus!

(For contrast, recall that there are no closed minimal immersions in $\mathbb{R}^3$.)

Outline



Stability of minimal immersions

The stability of minimal surfaces has attracted a lot of attention.

Recall that $|h|^2 = \kappa_1^2 + \kappa_2^2 = 4H^2 - 2K + 2c$, and let

$$J = -\left(\Delta + |h|^2 + 2c\right),$$

denote the L^2 self-adjoint Jacobi operator.

For a minimal immersion $\mathbf{r}$, we have

$$\delta^2\mathcal{A}(\mathbf{r})(u,u) = -\int_M u\Delta u + (4c-2K)u = \langle Ju, u\rangle_{L^2}.$$

This defines a bilinear operator,

$$\mathcal{I}_{\mathcal{A}}(\mathbf{r})(u,v) = \langle Ju, v\rangle_{L^2}, \qquad u, v \in C^\infty(\Sigma),$$

whose negative eigenvalues give the index of the immersion.

Classical results on $\mathcal{A}$-stability

- When $c > 0$, there are no stable, closed minimal surfaces.
- (do Carmo/Peng 1979) Stable, complete minimal surfaces in $\mathbb{R}^3$ are planes.
- (Barbosa/do Carmo/Eschenburg 1988) The only stable, closed CMC hypersurfaces in space forms are geodesic spheres.
- (Cheng/Tysk 1988) The only complete oriented minimal surface in R3 of index one with embedded ends is the catenoid.
- (Yang/Yau 1980) All others except the plane have index ≥ 2.

Willmore stability

On the other hand, we have

$$\delta^2\mathcal{W}(\mathbf{r})(u,u) = \int_M \frac{1}{2}(\Delta u)^2 + (3c - H^2 - 2K)u\Delta u + 2(H^2 - K + c)(4H^2 - K + 2c)u^2 + \int_M 2H\left(2u\langle h, \mathrm{Hess}\, u\rangle + h(\nabla u, \nabla u) + u\langle \nabla H, \nabla u\rangle - H|\nabla u|^2\right).$$

If $\mathbf{r}$ is minimal, this reduces to

$$\begin{aligned}\delta^2\mathcal{W}(\mathbf{r})(u,u) &= \frac{1}{2}\int_M (\Delta u + (4c - 2K)u)(\Delta u + (2c - 2K)u) \\ &= \frac{1}{2}\langle J(J + 2c)u, u\rangle_{L^2} := \langle Wu, u\rangle_{L^2},\end{aligned}$$

and the index operator is

$$\mathcal{I}_{\mathcal{W}}(\mathbf{r})(u,v) = \langle Wu, v\rangle_{L^2}, \qquad u, v \in C^\infty(\Sigma),$$

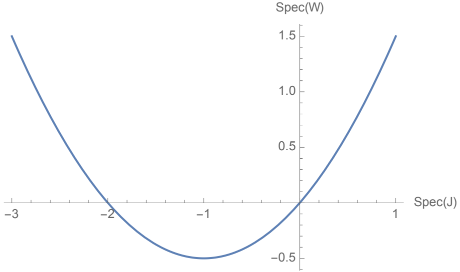
$\mathcal{A}$-stable implies $\mathcal{W}$-stable

Recall that minimal surfaces in S^3 are Willmore in $\mathbb{R}^3$ after stereographic projection.

For immersions in S^3, there is a nice relationship between $\mathcal{A}$-stable and $\mathcal{W}$-stable.

Proposition

Let $\mathbf{r} : \Sigma \to S^3$ be a minimal immersion. If $\mathbf{r}$ is $\mathcal{A}$-stable, then it is also $\mathcal{W}$-stable.



Unfortunately, *this statement is vacuous* in the case of closed Σ (Simons 1968 [1]).

What about the converse?

$\mathcal{W}$-stability is generally not enough for $\mathcal{A}$-stability.

1) The Clifford torus

$$T^2 = \{\frac{1}{\sqrt{2}}(e^{i\theta}, e^{i\phi}) \,|\, 0 \leq \theta, \phi < 2\pi\},$$

is flat and minimally embedded in S^3. Is it stable?

Its spectrum contains:

- $J = -4$, from constant functions.
- $J = -2$ from first eigenfunctions.

Therefore, T^2 is not $\mathcal{A}$-stable.

Conversely, $W = J(J+2)$ implies that no eigenvalues of W are negative, so T^2 is $\mathcal{W}$-stable.

Another example

Consider the equatorial $S^2 \subset S^3$. The eigenvalues of Δ are

$$\lambda_k = k(k+1).$$

Therefore, $\mathrm{Spec}\,(J)$ contains the value -2, so S^2 is not $\mathcal{A}$-stable.

On the other hand, $J = -2$ implies $W = 0$, so S^2 is $\mathcal{W}$-stable.

Note that S^2 is $\mathcal{A}$-stable among volume-preserving variations!

What is in the kernel?

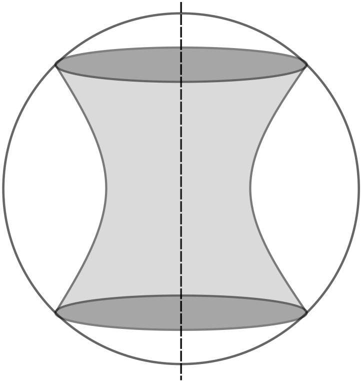
Let $\alpha \in \mathrm{Spec}(J)$. What variations of the minimal $\mathbf{r} : M \to S^3$ fix the Willmore energy to second order?

- If $\alpha = 0$, these are the area Jacobi fields (e.g. translations and rotations).
- If $\alpha = -2$, these are other Möbius transformations (e.g. "centered dilations" i.e. tangential projections to S^3 of constant vector fields on $\mathbb{R}^4$).

Determining all the "Jacobi fields" for minimal Willmore surfaces remains a significant open question!

In fact, almost nothing is known about the fields corresponding to $\alpha \in (-2, 0)$, even for specific surfaces.

Outline



The p-Willmore energy

It is also interesting to consider the **p-Willmore functional**

$$\mathcal{W}^p(\mathbf{r}) = \int_M \left(H^2 + c\right)^{\frac{p}{2}} \, d\mu_g, \qquad p \in \mathbb{R}.$$

Critical points of this energy are called **p-Willmore surfaces**, and have known similarities to minimal surfaces when $p > 2$.

For example, there are no closed p-Willmore surfaces immersed in $\mathbb{R}^3$, in view of the following result.

Theorem: G., Toda, Tran [2]

When $p > 2$, any p-Willmore surface $M \subset \mathbb{R}^3$ satisfying $H = 0$ on ∂M is minimal.

No scale-invariance for $p > 2$

Interestingly, this result is **not** true for $p = 2$. Examples of non-minimal Willmore catenoids satisfying $H = 0$ on the boundary were given in [3].

Though $\mathcal{W}^2$ is generally not conformally-invariant for surfaces with boundary, it is still invariant under uniform dilations.

Consider a dilation $\mathbf{r} \mapsto (1/t)\mathbf{r}$ for some $t > 0$. Then,

$$H \mapsto tH, \qquad d\mu_g \mapsto \frac{1}{t^2} d\mu_g,$$

and so the p-Willmore energy (in $\mathbb{R}^3$)

$$\mathcal{W}^p(\mathbf{r}) \mapsto t^{p-2}\mathcal{W}^p(\mathbf{r}),$$

which coincides (only!) when $p = 2$.

Question: Is scale-invariance responsible for the lack of non-minimal p-Willmore surfaces for $p > 2$?

Minimal p-Willmore surfaces in S^3

When $\mathbf{r} : M \to S^3$ is p-Willmore, it follows that

$$\begin{aligned}\delta^2\mathcal{W}^p(\mathbf{r})u &= \frac{p}{4}\int_\Sigma \left(\Delta u + (4-2K)\,u\right)\left(\Delta u + \left(4 - 2K - \tfrac{4}{p}\right)u\right) \\ &= \frac{p}{4}\left\langle J\left(J+\tfrac{4}{p}\right)u, u\right\rangle_{L^2}.\end{aligned}$$

Clearly, the immersion is $\mathcal{W}^p$-stable provided its Jacobi operator J has no eigenvalues in the interval $\left(\frac{-4}{p}, 0\right)$.

Proposition (G., Toda, Tran)

Any minimal immersion $\mathbf{r} : \Sigma \to S^3$ which is $\mathcal{W}^p$-stable for some value $p = p_0$ is also $\mathcal{W}^p$-stable for all $p > p_0$.

So, p-Willmore stability persists through increasing values of p!

Future work and open questions

A huge number of questions remain regarding the stability of $\mathcal{B}$-energy minimizers.

Even in the case of the Willmore energy, basic questions remain unanswered.

- There is (a lot of) evidence to support that the round sphere is the only stable genus 0 minimizer of $\mathcal{W}$. Is this true?
- There is (less) evidence to support that the Clifford torus is the only stable genus 1 minimizer of $\mathcal{W}$. Is this true?
- For minimal immersions into S^3 which are $\mathcal{W}$-unstable, what is the smallest value of p which makes them $\mathcal{W}^p$-stable?

Answers to these questions could help provide a simpler proof of the Willmore conjecture!

Thanks

Thank you!

References

J. Simons.
Minimal varieties in Riemannian manifolds.
Annals of Mathematics, pages 62–105, 1968.

A. Gruber, M. Toda, and H. Tran.
On the variation of curvature functionals in a space form with application to a generalized willmore energy.
Annals of Global Analysis and Geometry, May 2019.

Anna Dall'Acqua, Klaus Deckelnick, and Glen Wheeler.
Unstable willmore surfaces of revolution subject to natural boundary conditions.
Calculus of Variations and Partial Differential Equations, 48, 11 2013.