Curvature functionals and variational problems

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April 29, 2019

Biography and bibliography

Anthony Gruber: PhD candidate, 8/2015-8/2019.

Research interests: Differential geometry, computational geometry, geometric PDE, mathematical physics.

Source material

- A. Gruber, M. Toda, H. Tran, “On the variation of curvature functionals in a space form with application to a generalized Willmore energy”, *Annals of Global Analysis and Geometry* (to appear).
- A. Gruber, “Curvature functionals and p-Willmore energy” *PhD thesis*, defending 5/9/2019.
- E. Aulisa, A. Gruber, “Finite element models for the p-Willmore flow of surfaces” (in preparation).

Outline

Acknowledgements: (2) and (3) joint with M. Toda and H. Tran. (4) joint with E. Aulisa.

Examples

Functionals involving surface curvature have a range of applications across science and mathematics.

- Helfrich-Canham energy

$$E_H(M) := \int_M k_c(2H + c_0)^2 + \overline{k}K \, dS,$$

- Bulk free energy density

$$\sigma_F(M) = \int_M 2k(2H^2 - K) \, dS,$$

- Willmore energy

$$\mathcal{W}^2(M) = \int_M H^2 \, dS.$$

General bending energy

All these energies are special cases of a bending energy model proposed by Sophie Germain in 1820,

$$\mathcal{B}(M) = \int_M S(\kappa_1, \kappa_2)\, dS,$$

where S is a symmetric polynomial in the principal curvatures κ_1, κ_2.

By Newton's theorem, this is equivalent to the functional

$$\mathcal{F}(M) = \int_M \mathcal{E}(H, K)\, dS,$$

where $\mathcal{E}$ is smooth in $H = \frac{1}{2}(\kappa_1 + \kappa_2), K = \kappa_1 \kappa_2$.

Our problem

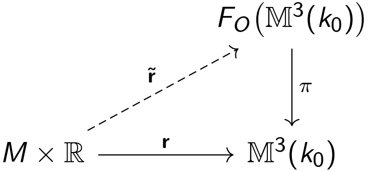
We study the functional $\mathcal{F}(M)$ on surfaces $M \subset \mathbb{M}^3(k_0)$ which are immersed in a *space form* of constant sectional curvature k_0.

Why leave Euclidean space?

- It's mathematically relevant (e.g. conformal geometry in S^3, geometry in projective space $\mathbb{C}P^3$).
- Physicists care about immersions in "Minkowski space", which has constant sectional curvature -1.
- Bending energy is different depending on the ambient space! For example, $(\kappa_1 - \kappa_2)^2 = 4(H^2 - K + k_0)$.

Geometric framework

Consider a variation of the surface M, i.e. a 1-parameter family of compactly supported immersions $\mathbf{r}(\mathbf{x}, t)$ as in the following diagram,



Choosing a section $\{\mathbf{e}_J\}$ of $F_O\big(\mathbb{M}^3(k_0)\big)$ and a dual basis $\{\omega^I\}$ such that $\omega^I(\mathbf{e}_J) = \delta^I_J$, it follows that:

- Metric on $\mathbb{M}^3(k_0)$: $\quad g = (\omega^1)^2 + (\omega^2)^2 + (\omega^3)^2$.
- Connection on $\mathbb{M}^3(k_0)$: $\quad \nabla \mathbf{e}_I = \mathbf{e}_J \otimes \omega^J_I$.
- Volume form on $\mathbb{M}^3(k_0) = \omega^1 \wedge \omega^2 \wedge \omega^3$.

Connection is Levi-Civita when $\omega^I_J = -\omega^J_I$.

Geometric framework (2)

The Cartan structure equations on $\mathbb{M}^3(k_0)$ are then

$$d\omega^I = -\omega^I_J \wedge \omega^J,$$
$$d\omega^I_J = -\omega^I_K \wedge \omega^K_J + \frac{1}{2} R^I_{JKL} \omega^K \wedge \omega^L.$$

We may assume the normal velocity of **r** satisfies

$$\frac{\partial \mathbf{r}}{\partial t} = u\,\mathbf{N},$$

for some smooth $u : M \times \mathbb{R} \to \mathbb{R}$. Pulling back the frame to $M \times \mathbb{R}$, we may further assume $\mathbf{e}_3 := \mathbf{N}$ is normal to $M \times \{t\}$ for each t, in which case

$$\overline{\omega}^i = \omega^i \qquad (i = 1, 2),$$
$$\overline{\omega}^3 = u\,dt.$$

Geometric framework (3)

Pulling back the structure equations on $\mathbb{M}^3(k_0)$, it is then possible to compute geometric information about M. In particular, $d(\overline{\omega}^3 - u\,dt) = 0$ implies

$$-\overline{\omega}^3_j \wedge \overline{\omega}^j - du \wedge dt = 0,$$

so that (by Cartan's Lemma) there are functions $u_1, u_2, \dot{u}$ and $h_{ij} = h_{ji}$ satisfying

$$\begin{bmatrix} \overline{\omega}^3_1 \\ \overline{\omega}^3_2 \\ du \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} & u_1 \\ h_{21} & h_{22} & u_2 \\ u_1 & u_2 & \dot{u} \end{bmatrix} \begin{bmatrix} \overline{\omega}_1 \\ \overline{\omega}_2 \\ dt \end{bmatrix}.$$

Since $\nabla \mathbf{N} = \mathbf{e}_j \otimes \overline{\omega}^j_3$, it follows that the h_{ij} are the components of the (symmetrized) second fundamental form of M,

$$\mathrm{II} = h_{ij}\,\overline{\omega}^1 \otimes \overline{\omega}^2 \otimes \mathbf{N}.$$

Geometric framework (4)

We can compute variations in a similar way. Pulling back the evolving functional through the inclusion $\iota_t : M \to M \times \{t\}$, we differentiate

$$\delta\mathcal{F}(M) = \frac{d}{dt}\bigg|_{t=0} \int_{M\times\{t\}} \mathcal{E}(H,K)\,\overline{\omega}^1 \wedge \overline{\omega}^2 = \int_M \mathcal{L}_{\partial/\partial t}\big(\mathcal{E}(H,K)\,\overline{\omega}^1 \wedge \overline{\omega}^2\big).$$

By Cartan's formula, this becomes

$$\delta\mathcal{F}(M) = \int_M \frac{\partial}{\partial t} \lrcorner\, d\big(\mathcal{E}(H,K)\,\overline{\omega}^1 \wedge \overline{\omega}^2\big),$$

since $\frac{\partial}{\partial t} \lrcorner\, \overline{\omega}^1 \wedge \overline{\omega}^2 = 0$.

General first variation

There is then the following necessary condition for criticality.

Theorem: G., Toda, Tran

The first variation of the curvature functional $\mathcal{F}$ is given by

$$\delta \int_M \mathcal{E}(H,K)\, dS$$
$$= \int_M \left(\frac{1}{2}\mathcal{E}_H + 2H\mathcal{E}_K\right)\Delta u + \left((2H^2 - K + 2k_0)\mathcal{E}_H + 2HK\mathcal{E}_K - 2H\mathcal{E}\right)u$$
$$- \mathcal{E}_K \langle h, \mathrm{Hess}\, u\rangle \; dS,$$

where $\mathcal{E}_H, \mathcal{E}_K$ denote the partial derivatives of $\mathcal{E}$ with respect to H resp. K, and h is the shape operator of M ($\mathbf{II} = h\,\mathbf{N}$).

General second variation

Theorem: G., Toda, Tran

At a critical immersion of M, the second variation of $\mathcal{F}$ is given by

$$\delta^2 \int_M \mathcal{E}(H,K)\, dS = \int_M \left(\frac{1}{4}\mathcal{E}_{HH} + 2H\mathcal{E}_{HK} + 4H^2\mathcal{E}_{KK} + \mathcal{E}_K\right)(\Delta u)^2\, dS$$

$$+ \int_M \mathcal{E}_{KK}\langle h, \text{Hess}\, u\rangle^2\, dS - \int_M (\mathcal{E}_{HK} + 4H\mathcal{E}_{KK})\Delta u \langle h, \text{Hess}\, u\rangle\, dS$$

$$+ \int_M \mathcal{E}_K\left(u\langle\nabla K, \nabla u\rangle - 3u\langle h^2, \text{Hess}\, u\rangle - 2\, h^2(\nabla u, \nabla u) - |\text{Hess}\, u|^2\right) dS$$

$$+ \int_M \Big((2H^2 - K + 2k_0)\mathcal{E}_{HH} + 2H(4H^2 - K + 4k_0)\mathcal{E}_{HK} + 8H^2K\mathcal{E}_{KK}$$

$$- 2H\mathcal{E}_H + (3k_0 - K)\mathcal{E}_K - \mathcal{E}\Big) u\Delta u\, dS$$

$$+ \int_M \Big((2H^2 - K + 2k_0)^2\mathcal{E}_{HH} + 4HK(2H^2 - K + 2k_0)\mathcal{E}_{HK} + 4H^2K^2\mathcal{E}_{KK}$$

$$- 2K(K - 2k_0)\mathcal{E}_K - 2HK\mathcal{E}_H + 2(K - 2k_0)\mathcal{E}\Big) u^2\, dS$$

$$+ \int_M (2\mathcal{E}_H + 6H\mathcal{E}_K - 2(2H^2 - K + 2k_0)\mathcal{E}_{HK} - 4HK\mathcal{E}_{KK})u\langle h, \text{Hess}\, u\rangle\, dS$$

$$+ \int_M (\mathcal{E}_H + 4H\mathcal{E}_K)h(\nabla u, \nabla u)\, dS + \int_M \mathcal{E}_H\, u\langle\nabla H, \nabla u\rangle\, dS$$

$$- \int_M (2(K - k_0)\mathcal{E}_K + H\mathcal{E}_H)|\nabla u|^2\, dS,$$

where the subscripts $\mathcal{E}_{HH}, \mathcal{E}_{HK}, \mathcal{E}_{KK}$ denote the second partial derivatives of $\mathcal{E}$ in the appropriate variables.

Advantages of these variational results

- Valid in any space form of constant sectional curvature k_0, not only Euclidean space.
- Quantities involved are as elementary as possible – directly computable from surface fundamental forms for a given u.

Very useful also for studying specific functionals. For example, these expressions give immediately the known variation of the Willmore functional,

$$\delta \int_M H^2\, dS = \int_M \left(H\Delta u + 2H(H^2 - K + 2k_0)u \right) dS.$$

Note that it follows that closed Willmore surfaces in $\mathbb{M}^3(k_0)$ are characterized by the equation

$$\Delta H + 2H(H^2 - K + 2k_0) = 0.$$

The p-Willmore energy

Another interesting curvature functional is the p-Willmore energy,

$$\mathcal{W}^p(M) = \int_M H^p \, dS \qquad (p \geq 1)$$

- Natural from the point of view of bending energies.
- Generalizes the usual Willmore energy.
- Not conformally invariant when $p \neq 2$.
- Also encompasses the total mean curvature functional $\mathcal{W}^1$, and can be extended to consider the area functional as $\mathcal{W}^0$.

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- Highly connected to minimal surface theory when $p > 2$!!

Variations of p-Willmore energy

Corollary: G., Toda, Tran

The first variation of $\mathcal{W}^p$ is given by

$$\delta \int_M H^p \, dS = \int_M \left[\frac{p}{2} H^{p-1} \Delta u + (2H^2 - K + 2k_0) p H^{p-1} u - 2H^{p+1} u\right] dS,$$

Moreover, the second variation of $\mathcal{W}^p$ at a critical immersion is given by

$$\begin{aligned}
\delta^2 \int_M H^p \, dS &= \int_M \frac{p(p-1)}{4} H^{p-2} (\Delta u)^2 \, dS \\
&+ \int_M p H^{p-1} \left(h(\nabla u, \nabla u) + 2u \langle h, \mathrm{Hess}\, u \rangle + u \langle \nabla H, \nabla u \rangle - H |\nabla u|^2 \right) dS \\
&+ \int_M \left((2p^2 - 4p - 1) H^p - p(p-1) K H^{p-2} + 2p(p-1) k_0 H^{p-2} \right) u \Delta u \, dS \\
&+ \int_M \Big(4p(p-1) H^{p+2} - 2(p-1)(2p+1) K H^p + p(p-1) K^2 H^{p-2} \\
&\quad + 4(2p^2 - 2p - 1) k_0 H^p - 4p(p-1) k_0 K H^{p-2} + 4p(p-1) k_0^2 H^{p-2} \Big) u^2 \, dS.
\end{aligned}$$

Connection to minimal surfaces

Define a **p-Willmore surface** to be any M satisfying the Euler-Lagrange equation,

$$\frac{p}{2}\Delta H^{p-1} - p(2H^2 - K + 2k_0)H^{p-1} + 2H^{p+1} = 0 \quad \text{on } M.$$

Then, it is possible to show the following:

Theorem: G., Toda, Tran

When $p > 2$, any p-Willmore surface $M \subset \mathbb{R}^3$ satisfying $H = 0$ on ∂M is minimal.

More precisely, let $p > 2$ and $\mathbf{R} : M \to \mathbb{R}^3$ be an immersion of the p-Willmore surface M with boundary ∂M. If $H = 0$ on ∂M, then $H \equiv 0$ everywhere on M.

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$(p > 2)$-Willmore surface with $H = 0$ on $\partial M \iff$ minimal surface!

Sketch of proof

Let $\mathbf{n}$ be conormal to the immersion $\mathbf{R}$ on ∂M. The first goal is to establish the following integral equality, inspired by a result from Bergner et al. [1]:

$$\int_{\partial M} \nabla_{\mathbf{n}}(H^{p-1})\langle \mathbf{R}, \mathbf{N}\rangle = \int_{\partial M} H^{p-1}\big(\langle \nabla_{\mathbf{n}}\mathbf{N}, \mathbf{R}\rangle + (2/p)H\langle \nabla_{\mathbf{n}}\mathbf{R}, \mathbf{R}\rangle\big) + \frac{2(p-2)}{p}\int_M H^p.$$

- Similar computations as before establish the geometric identities

$$\Delta\mathbf{R} - 2H\mathbf{N} = 0,$$
$$\Delta\mathbf{N} + 2\nabla_{\nabla H}\mathbf{R} + 2(2H^2 - K)\mathbf{N} = 0.$$

- On a p-Willmore surface,

$$-\int_M \big(pH^{p-1}(2H^2-K) - 2H^{p+1}\big)\langle \mathbf{R}, \mathbf{N}\rangle = \frac{p}{2}\int_M \Delta(H^{p-1})\langle \mathbf{R}, \mathbf{N}\rangle$$
$$= \int_M H^{p-1}\Delta\langle \mathbf{R}, \mathbf{N}\rangle + \text{boundary terms}.$$

Sketch of proof (2)

Continuing this computation eventually yields

$$
\begin{aligned}
&-\int_M \left(pH^{p-1}(2H^2-K)-2H^{p+1}\right)\langle \mathbf{R},\mathbf{N}\rangle + \text{boundary terms} \\
&= \int_M (2-p)H^p - \int_M \left(pH^{p-1}(2H^2-K)-2H^{p+1}\right)\langle \mathbf{R},\mathbf{N}\rangle,
\end{aligned}
$$

and algebraic manipulations yield the result.

Now, when $p>2$ and $H=0$ on ∂M, it follows that

$$0=\int_{\partial M}\nabla_{\mathbf{n}}(H^{p-1})\langle \mathbf{R},\mathbf{N}\rangle = \frac{2(p-2)}{p}\int_M H^p.$$

When p is even, it follows immediately that $H=0$ a.e. on M, so $H\equiv 0$. When p is odd (or not integer), divide M into regions where $H>0$ and $H<0$. Continuity implies that $H=0$ on the boundaries, so the above integral equality applies. Conclude $H\equiv 0$ everywhere on M.

Consequences

This result has a number of interesting consequences. First,

- NOT true for $p = 2$: many solutions (non-minimal catenoids, etc.) to Willmore equation with $H = 0$ on boundary.

Further, it follows immediately that

Corollary: G., Toda, Tran

There are no closed p-Willmore surfaces immersed in $\mathbb{R}^3$ when $p > 2$.

Proof. There are no closed minimal surfaces in $\mathbb{R}^3$.

This means,

- The round sphere, Clifford torus, etc. are no longer minimizing in general for $\mathcal{W}^p$.
- Minimization must be modified if there are to be closed solutions for all p.

Volume-constrained p-Willmore

Since $\mathcal{W}^p$ is physically motivated as a bending energy model, it is reasonable to consider its minimization subject to geometric constraints.

Let $M = \partial D$ and recall the volume functional

$$\mathcal{V} = \int_D dV = \int_{M\times[0,t]} \mathbf{R}^*(dV),$$

with first variation

$$\delta\mathcal{V} = \int_M u\, dS.$$

So, (by a Lagrange multiplier argument) M is a **volume-constrained p-Willmore surface** provided there is a constant C such that

$$\frac{p}{2}\Delta H^{p-1} - p(2H^2 - K + 2k_0)H^{p-1} + 2H^{p+1} = C.$$

Volume-constrained p-Willmore (2)

Why a volume constraint?

- Mimics the behavior of a lipid membrane in a solution with varying concentrations of solute.
- Acts as a "substitute" for conformal invariance in the sense that it naturally limits the space of allowable surfaces.
- Allows for certain closed surfaces to be at least "almost stable". Note the following result for spheres.

Theorem: G., Toda, Tran

The round sphere $S^2(r)$ immersed in Euclidean space is **not** a stable local minimum of $\mathcal{W}^p$ under general volume-preserving deformations for each $p > 2$. More precisely, the bilinear index form is negative definite on the eigenspace of the Laplacian associated to the first eigenvalue, and it is positive definite on the orthogonal complement subspace.

Sketch of proof – sphere is unstable

Note the first and second variations of $\mathcal{W}^p\big(S^2(r)\big)$:

$$\delta\mathcal{W}^p\big(S^2(r)\big) = \frac{p-2}{r^{p+1}}\int_{S^2(r)} u\, dS,$$

$$\delta^2\mathcal{W}^p\big(S^2(r)\big) = \frac{1}{r^p}\int_{S^2(r)}\left(\frac{p(p-1)r^2}{4}(\Delta u)^2 + (p^2-p-1)u\Delta u + \frac{(p-1)(p-2)}{r^2}u^2\right) dS.$$

- Notice the first variation vanishes for volume-preserving u.
- At a first eigenfunction of Δ on $S^2(r)$, we have $\Delta u + (2/r^2)u = 0$, in which case

$$\delta^2\mathcal{W}^p\big(S^2(r)\big) = \frac{1}{r^{p+2}}\int_{S^2(r)} 2u^2(2-p)\, dS < 0, \qquad p > 2.$$

- Conclude that the sphere is unstable when $p > 2$.

Sketch of proof – cause of instability

Note the Poincare' inequality due to Elliott et al. [2]:

Lemma: Elliott, Fritz, Hobbes

For any smooth nonconstant function $u : S^2(r) \to \mathbb{R}$ such that $u \in \{v : \Delta v = -(2/r^2)v\}^{\perp}$,

$$\int_{S^2(r)} u^2\, dS \leq \frac{r^2}{6} \int_{S^2(r)} |\nabla u|^2\, dS \leq \frac{r^4}{36} \int_{S^2(r)} (\Delta u)^2\, dS.$$

Using this, it follows that the index form satisfies

$$\begin{aligned} I_{\mathcal{W}^p}(S^2(r))(u,u) = \delta^2 \int_{S^2(r)} H^p\, dS &\geq \int_{S^2(r)} \frac{2p^2-3p+4}{2r^2}\, u^2\, dS \\ &\geq C(p,r) \int_{S^2(r)} u^2\, dS, \end{aligned}$$

for all allowed values of p.

Computational modeling

Can also study p-Willmore surfaces in $\mathbb{R}^3$ experimentally by writing appropriate weak-form equations which can be discretized using finite elements.

Differences from the theoretical setting:

- Cannot choose a preferential frame in which to calculate the variations.
- Must consider general variations, which may have tangential as well as normal components.
- Convenient to work with the identity map $u : M \to M$ on the surface, not directly with surface immersions.

In particular, we will model the p-Willmore flow equation

$$\dot{u} = -\delta \mathcal{W}^p(u).$$

Computational modeling (2)

Problem: Closed p-Willmore flow with volume and area constraint

Let $p \geq 2$, $Y = 2HN$, and $W := (Y \cdot N)^{p-2} Y$. Determine a family $M(t)$ of closed surfaces with identity maps $u(X,t)$ such that $M(0)$ has initial volume V_0 and surface area A_0, and the equation

$$\dot{u} = \delta \left(\mathcal{W}^p + \lambda \mathcal{V} + \mu \mathcal{A} \right),$$

is satisfied for all $t \in (0, T]$ and for some piecewise-constant functions λ, μ. Equivalently, if $M(t)$ is the image of the immersion $X(t)$, find functions u, Y, W, λ, μ on $M(t)$ such that the equations

$$\int_M \dot{u} \cdot \varphi + \lambda(\varphi \cdot N) + \mu \nabla_M u : \nabla_M \varphi + \left((1-p)(Y \cdot N)^p - p \nabla_M \cdot W \right) \nabla_M \cdot \varphi$$
$$+ p D(\varphi) \nabla_M u : \nabla_M W - p \nabla_M \varphi : \nabla_M W = 0,$$
$$\int_M Y \cdot \psi + \nabla_M u : \nabla_M \psi = 0,$$
$$\int_M W \cdot \xi - (Y \cdot N)^{p-2} Y \cdot \xi = 0,$$
$$\int_M 1 = A_0,$$
$$\int_M u \cdot N = V_0,$$

are satisfied for all $t \in (0, T]$ and all $\varphi, \psi, \xi \in H_0^1(M(t))$.

Computational modeling (3)

Note that the p-Willmore energy always decreases along the p-Willmore flow.

Theorem: Aulisa, G.

The (unconstrained) closed surface p-Willmore flow is energy decreasing for $p \geq 2$, i.e.

$$\int_{M(t)} |\dot{u}|^2 + \frac{d}{dt} \int_{M(t)} (Y \cdot N)^p = 0,$$

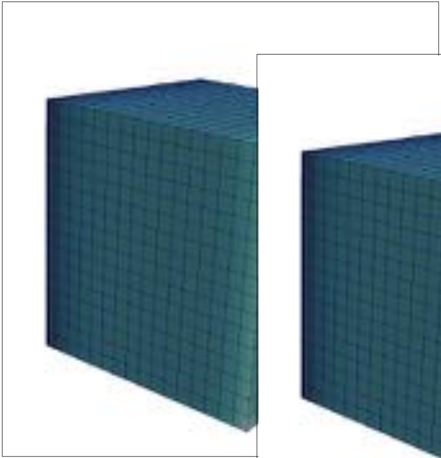
for all $t \in (0, T]$.

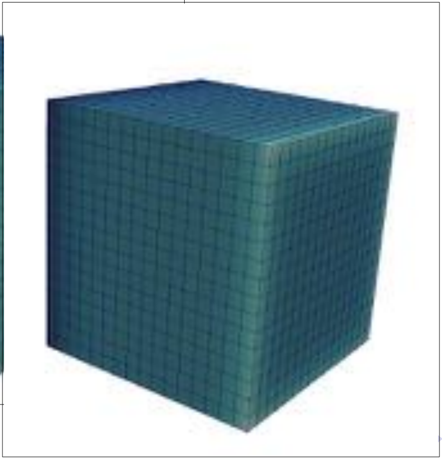
- This is GOOD when p is even, since energy is bounded.
- When p is odd, stability is highly dependent on initial energy configuration.

Conjecture for odd p: A flow started from a surface where $\mathcal{W}^p > 0$ remains so for all time.

Results: Cube

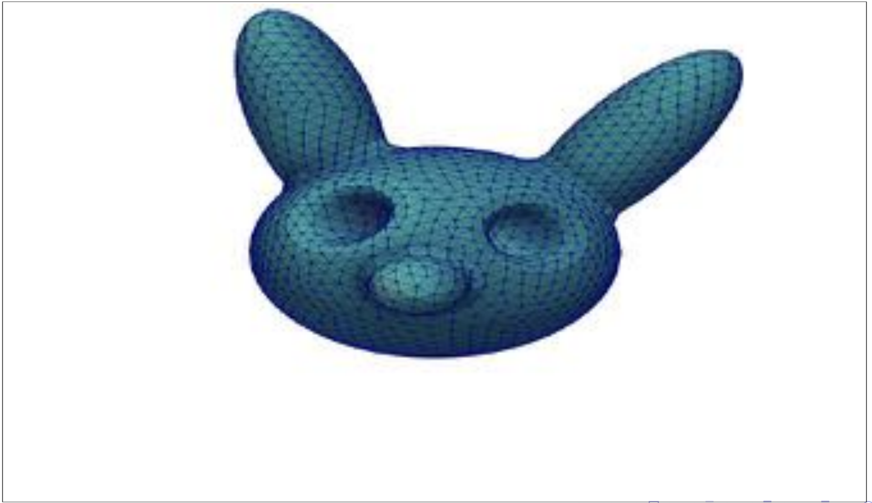
Willmore evolution of a cube with volume constraint (left) and unconstrained (right).





Results: Dog

The 3-Willmore evolution of a genus 0 dog mesh constrained by enclosed volume. Note the initial 3-Willmore energy is positive.



Results: Knot

The Willmore evolution of a trefoil knot constrained by surface area and enclosed volume.

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