A conformally-adjusted Willmore flow of closed surfaces

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Biography and bibliography

Anthony Gruber: PhD candidate, 8/2015-8/2019.

Research interests: Differential geometry, computational geometry, geometric PDE, mathematical physics.

Source material

- A. Gruber, "Curvature functionals and p-Willmore energy" *PhD thesis*, defending 5/9/2019.
- E. Aulisa, A. Gruber, "Finite element models for the p-Willmore flow of surfaces" (in preparation).
- A. Gruber, M. Toda, H. Tran, "On the variation of curvature functionals in a space form with application to a generalized Willmore energy", *Annals of Global Analysis and Geometry* (to appear).

Outline

Acknowledgements: All original results in this seminar are joint with E. Aulisa unless otherwise stated (some results with M. Toda and H. Tran).

The Willmore energy

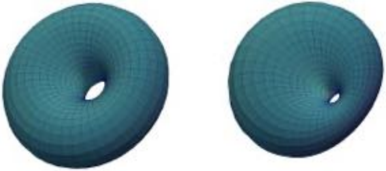
Let $\mathbf{R} : M \to \mathbb{R}^3$ be a smooth immersion of the closed surface M. Then, the Willmore energy is defined as

$$\mathcal{W}(M) = \int_M H^2 \, dS,$$

where $H = (1/2)(\kappa_1 + \kappa_2)$ is the mean curvature of the surface.

Facts:

- Critical points of $\mathcal{W}(M)$ are called Willmore surfaces, and arise as natural generalizations of minimal surfaces.
- $\mathcal{W}(M)$ is invariant under reparametrizations, and less obviously under *conformal transformations of the ambient metric* (Mobius transformations of $\mathbb{R}^3$)



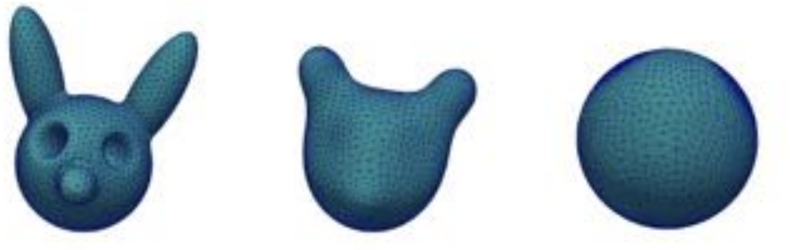
The Willmore energy (2)

From an aesthetic perspective, the Willmore energy encourages surface fairing (i.e. smoothing). How to see this?

$$\frac{1}{4}\int_M (\kappa_1 - \kappa_2)^2 \, dS = \int_M (H^2 - K)\, dS = \mathcal{W}(M) - 2\pi\chi(M),$$

by the Gauss-Bonnet theorem.

Conclusion: The Willmore energy punishes surfaces for being non-umbilic!



Examples of Willmore-type energies

The Willmore energy arises frequently in mathematical biology, physics and computer vision – sometimes under different names.

- Helfrich-Canham energy,

$$E_H(M) := \int_M k_c(2H + c_0)^2 + \overline{k}K\, dS,$$

- Bulk free energy density,

$$\sigma_F(M) = \int_M 2k(2H^2 - K)\, dS,$$

- Surface torsion,

$$\mathcal{S}(M) = \int_M 4(H^2 - K)\, dS$$

When M is closed, all share critical surfaces with $\mathcal{W}(M)$!

The p-Willmore energy

To model the Willmore flow, we need a good expression for the first variation of the Willmore energy. More generally, we will consider a generalization called the *p-Willmore energy*,

$$\mathcal{W}^p(M) = \int_M H^p\, dS, \qquad p \in \mathbb{Z}_{\geq 0}.$$

Notice that the Willmore energy is recovered as $\mathcal{W}^2$.

Why generalize Willmore?

- Conformal invariance is beautiful but very un-physical: unnatural for bending energy.
- $\mathcal{W}^0, \mathcal{W}^1$, and $\mathcal{W}^2$ are quite different. Are other $\mathcal{W}^p$ different?

Theorem: G., Toda, Tran

When $p > 2$, any p-Willmore surface $M \subset \mathbb{R}^3$ satisfying $H = 0$ on ∂M is minimal.

More precisely, let $p > 2$ and $\mathbf{R} : M \to \mathbb{R}^3$ be an immersion of the p-Willmore surface M with boundary ∂M. If $H = 0$ on ∂M, then $H \equiv 0$ everywhere on M.

The variational framework

This framework is due to Dziuk and Elliott [1]. Consider a parametrization $X_0 : V \subset \mathbb{R}^2 \to \mathbb{R}^3$ of (a portion of) the surface M, and let $u_0 : M \to \mathbb{R}^3$ be identity on M, so $u \circ X = X$.

A variation of M is a smooth function $\varphi : M \to \mathbb{R}^3$ and a 1-parameter family $u(x,t) : M \times (-\varepsilon, \varepsilon) \to \mathbb{R}^3$ such that $u(x,0) = u_0$ and

$$u(x,t) = u_0(x) + t\varphi(x).$$

Note that this pulls back to a variation $X : V \times (-\varepsilon, \varepsilon) \to \mathbb{R}^3$,

$$X(v,t) = X_0(v) + t\Phi(v),$$

where $\Phi = \varphi \circ X$. Note further that (since u is identity on $X(t)$) the time derivatives are related by

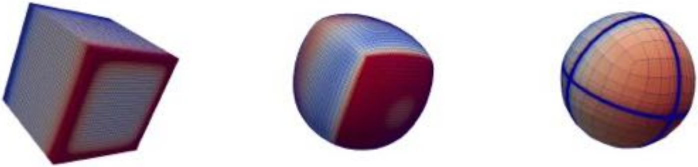
$$\dot{u} = \frac{d}{dt}u(X,t) = \nabla u \cdot \dot{X} + u_t = \dot{X}.$$

Computational challenges

There are notable differences here from the purely theoretical setting:

- Cannot choose a preferential frame in which to calculate derivatives; no natural adaptation (e.g. moving frame) is possible.
- Must consider general variations φ, which may have tangential as well as normal components.
- Must avoid geometric terms that are not easily discretized, such as K and $\nabla_M N$.

Can have very irritating *mesh sliding*:



Later, we will see a fix for this!

Calculating the first variation

Our goal is now to find a weak-form expression for the p-Willmore flow equation,

$$\dot{u} = -\delta\,\mathcal{W}^p.$$

First, note that the components of the induced metric on M are

$$g_{ij} = \partial_{x_i} X \cdot \partial_{x_j} X = X_i \cdot X_j$$

so that the surface gradient of a function f defined on M can be expressed as

$$(\nabla_M f) \circ X = g^{ij} X_i F_j,$$

where $F = f \circ X$ is the pullback of f through the parametrization X, and $g^{ik} g_{kj} = \delta^i_j$.

The Laplace-Beltrami operator on M is then

$$(\Delta_M f) \circ X = (\nabla_M \cdot \nabla_M f) \circ X = \frac{1}{\sqrt{\det g}} \partial_j \left(\sqrt{\det g}\, g^{ij} F_i \right).$$

Calculating the first variation (2)

Let $Y := \Delta u = 2HN$ be the mean curvature vector of $M \subset \mathbb{R}^3$. Then, the p-Willmore functional (modulo a factor of 2^p) can be expressed as

$$\mathcal{W}^p(M) = \int_M (Y \cdot N)^p.$$

It is then relatively straightforward to compute the p-Willmore Euler-Lagrange equation

$$\frac{p}{2}\Delta_M (Y \cdot N)^{p-1} - p|\nabla_M N|^2 (Y \cdot N)^{p-1} + \frac{1}{2}(Y \cdot N)^{p+1} = 0,$$

for a normal variation of $\mathcal{W}^p$.

Challenges:

- Express this 4^{th} order PDE weakly.
- Include the possibility of tangential motion.
- Suppress derivatives of the vector N.

Possible with some clever rearrangement and a splitting technique applied by G. Dziuk in [2].

Finding a weak formulation

To split the p-Willmore equation into a weak form system of 2^{nd} order PDE, first notice that $Y = \Delta_M u$ implies

$$\int Y \cdot \psi + \nabla_M u : \nabla_M \psi = 0, \qquad \forall \psi \in H_0^1(M).$$

Letting $W := (Y \cdot N)^{p-2} Y$ and $D(\varphi) := \nabla_M \varphi + (\nabla_M \varphi)^T$, a significant amount of computation then yields the p-Willmore flow system ($\varphi, \psi, \xi \in H_0^1(M)$)

$$\begin{aligned}
\int_M \dot{u} \cdot \varphi + \big((1-p)(Y \cdot N)^p - p \nabla_M \cdot W\big) \nabla_M \cdot \varphi \\
+ pD(\varphi) \nabla_M u : \nabla_M W - p \nabla_M \varphi : \nabla_M W = 0, \\
\int_M Y \cdot \psi + \nabla_M u : \nabla_M \psi = 0, \\
\int_M W \cdot \xi - (Y \cdot N)^{p-2} Y \cdot \xi = 0,
\end{aligned}$$

which is a weak formulation of $\dot{u} = -\delta \mathcal{W}^p$.

Properties of the p-Willmore flow: energy decrease

Theorem: Aulisa, G.

The (unconstrained) closed surface p-Willmore flow is energy decreasing for integer $p \geq 2$, i.e.

$$\int_{M(t)} |\dot{u}|^2 + \frac{d}{dt}\int_{M(t)} (Y \cdot N)^p = 0,$$

for all $t \in (0, T]$.

- This is GOOD when p is even, since energy is bounded from below.
- When p is odd, stability is highly dependent on initial energy configuration.

Conjecture for odd p: The p-Willmore flow started from a surface where $\mathcal{W}^p > 0$ remains ≥ 0 for all time. (Suggested by simulation)

The p-Willmore flow problem

Problem: Closed p-Willmore flow with volume and area constraint

Let $p \geq 2$, $Y = 2HN$, and $W := (Y \cdot N)^{p-2} Y$. Determine a family $M(t)$ of closed surfaces with identity maps $u(X, t)$ such that $M(0)$ has initial volume V_0, initial surface area A_0, and the equation

$$\dot{u} = \delta\,(\mathcal{W}^p + \lambda \mathcal{V} + \mu \mathcal{A}),$$

is satisfied for all $t \in (0, T]$ and for some piecewise-constant functions λ, μ.

Equivalently, find functions u, Y, W, λ, μ on $M(t)$ such that the equations

$$\int_M \dot{u} \cdot \varphi + \lambda(\varphi \cdot N) + \mu \nabla_M u : \nabla_M \varphi + \big((1-p)(Y \cdot N)^p - p \nabla_M \cdot W\big) \nabla_M \cdot \varphi$$

$$+ pD(\varphi) \nabla_M u : \nabla_M W - p \nabla_M \varphi : \nabla_M W = 0,$$

$$\int_M Y \cdot \psi + \nabla_M u : \nabla_M \psi = 0,$$

$$\int_M W \cdot \xi - (Y \cdot N)^{p-2} Y \cdot \xi = 0,$$

$$\int_M 1 = A_0,$$

$$\int_M u \cdot N = V_0,$$

are satisfied for all $t \in (0, T]$ and all $\varphi, \psi, \xi \in H_0^1(M(t))$.

How do we implement this? Algorithm:

Let $\tau > 0$ be a fixed step-size and $u^k := u(\cdot, k\tau)$. The p-Willmore flow algorithm proceeds as follows:

1. Given the initial surface position u_h^0, generate the initial curvature data Y_h^0, W_h^0 by solving

$$\int_{M_h^0} Y_h^0 \cdot \psi_h + \nabla_{M_h^0} u_h^0 : \nabla_{M_h^0} \psi_h = 0,$$
$$\int_{M_h^0} W_h^0 \cdot \xi_h - (Y_h^0 \cdot N_h^0)^{p-2} Y_h^0 \cdot \xi_h = 0.$$

Algorithm: p-Willmore flow loop

2. For integer $0 \leq k \leq T/\tau$, flow the surface according to the following procedure:
   1. Solve the (discretized) weak form equations: obtain the positions $\tilde{u}_h^{k+1}$, curvatures $\tilde{Y}_h^{k+1}$ and $\tilde{W}_h^{k+1}$, and Lagrange multipliers λ_h^{k+1} and μ_h^{k+1}.
   2. Minimize conformal distortion of the surface mesh $\tilde{u}_h^{k+1}$, yielding new positions u_h^{k+1}.
   3. Compute the updated curvature information Y_h^{k+1} and W_h^{k+1} from u_h^{k+1}.
3. Repeat step 2 until the desired time T.

Conformal correction step 2.2: idea

To correct mesh sliding at each time step, the goal is to enforce the "Cauchy-Riemann equations" on the tangent bundle TM.

Let $X : V \to \operatorname{Im} \mathbb{H}$ be an immersion of M, and J be a complex structure (rotation operator $J^2 = -\mathrm{Id}_{TV}$) on TV. Then, if $*\alpha = \alpha \circ J$ is the usual Hodge star on forms,

Thm: Kamberov, Pedit, Pinkall [3]

X is conformal iff there is a Gauss map $N : M \to \operatorname{Im} \mathbb{H}$ such that $*dX = N\, dX$.

Note that,

- $N \perp dX(v)$ for all tangent vectors $v \in TV$.
- $v, w \in \operatorname{Im} \mathbb{H} \longrightarrow vw = -v \cdot w + v \times w$.

Conclusion: conformality may be enforced by requiring $*dX(v) = N \times dX(v)$ on a basis for TV!

Conformal correction step 2.2: implementation

Choose x^1, x^2 as coordinates on V, then:

- $\partial_1 := \partial_{x^1}$ and $\partial_2 := \partial_{x^2}$ are a basis for TV.
- $dX(\partial_1) := X_1$ and $dX(\partial_2) := X_2$ are a basis for TM.
- $J\partial_1 = \partial_2$, $J\partial_2 = -\partial_1$.
- $\nabla_{dX(v)}u = \nabla_v X$ on M.

Instead of enforcing conformality explicitly, we minimize an energy functional. First, define

$$\mathcal{CD}_v(u) = \frac{1}{2}\int_M |\nabla_{dX(Jv)}u - N \times \nabla_{dX(v)}u|^2 = \frac{1}{2}\int_M |\nabla_{Jv}X - N \times \nabla_v X|^2.$$

Standard minimization techniques lead to the necessary condition

$$\delta\,\mathcal{CD} = \int_M \left(\nabla_{dX(Jv)}u - N \times \nabla_{dX(v)}u\right) \cdot \left(\nabla_{dX(Jv)}\varphi - N \times \nabla_{dX(v)}\varphi\right) = 0.$$

Conformal correction step 2.2: implementation (2)

So, choosing the basis $\{X_1, X_2\}$ for TM, we enforce

$$\int_M (\nabla_{X_2} u - N \times \nabla_{X_1} u) \cdot (\nabla_{X_2} \varphi - N \times \nabla_{X_1} \varphi) + \int_M (\nabla_{X_1} u + N \times \nabla_{X_2} u) \cdot (\nabla_{X_1} \varphi + N \times \nabla_{X_2} \varphi) = 0.$$

Important: To ensure this "reparametrization" does not undo the Willmore flow, we use a Lagrange multiplier ρ to move only on TM.

Specifically, if $\nabla_{M_{h,i}} = \nabla_{M_{h,X_i}}$, we solve for u_h^{k+1}, ρ_h^{k+1} satisfying

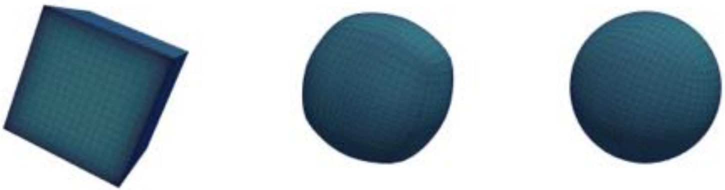
$$\int_{M_h^k} \rho_h^{k+1}(\varphi_h \cdot N_h^k) + (\nabla_{M_{h,2}^k} u_h^{k+1} - N_h^k \times \nabla_{M_{h,1}^k} u_h^{k+1}) \cdot (\nabla_{M_{h,2}^k} \varphi_h - N_h^k \times \nabla_{M_{h,1}^k} \varphi_h) + (\nabla_{M_{h,1}^k} u_h^{k+1} + N_h^k \times \nabla_{M_{h,2}^k} u_h^{k+1}) \cdot (\nabla_{M_{h,1}^k} \varphi_h + N_h^k \times \nabla_{M_{h,2}^k} \varphi_h) = 0,$$

$$\int_{M_h^k} (u_h^{k+1} - \tilde{u}_h^{k+1}) \cdot N_h^k = 0.$$

Conformal correction step 2.2: notes

This conformal correction is important because:

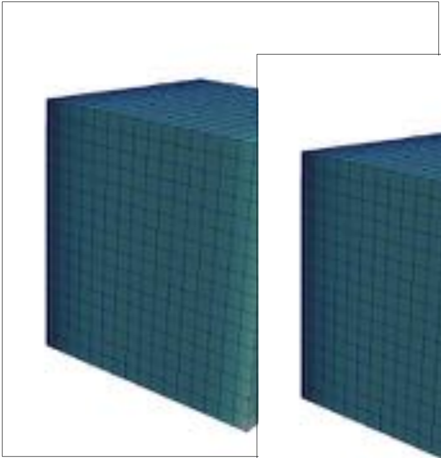
- Dramatically improves mesh quality during the p-Willmore flow.
- Keeps simulation from breaking due to mesh degeneration.
- Mitigates the artificial barrier to flow continuation caused by a bad mesh.

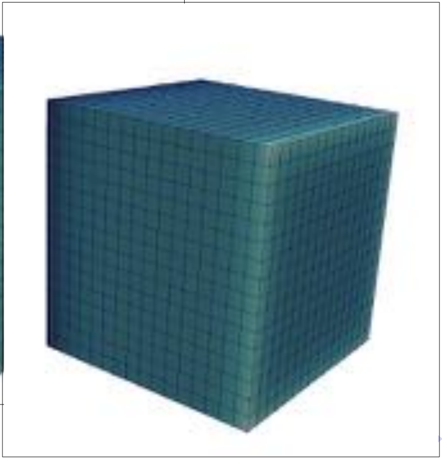


WARNING: This procedure only makes sense for quadrilateral meshes! Edges of a triangle cannot be mutually orthogonal.

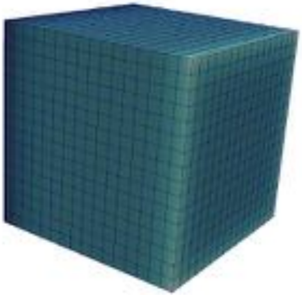
Results: constrained vs unconstrained Willmore

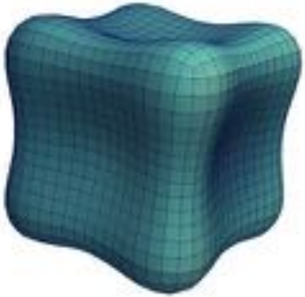
Willmore evolution of a cube with volume constraint (left) and unconstrained (right).





The Willmore flow of a cube. By constraining both surface area and enclosed volume, the cube can no longer flow to a sphere, creating a different minimizing surface.

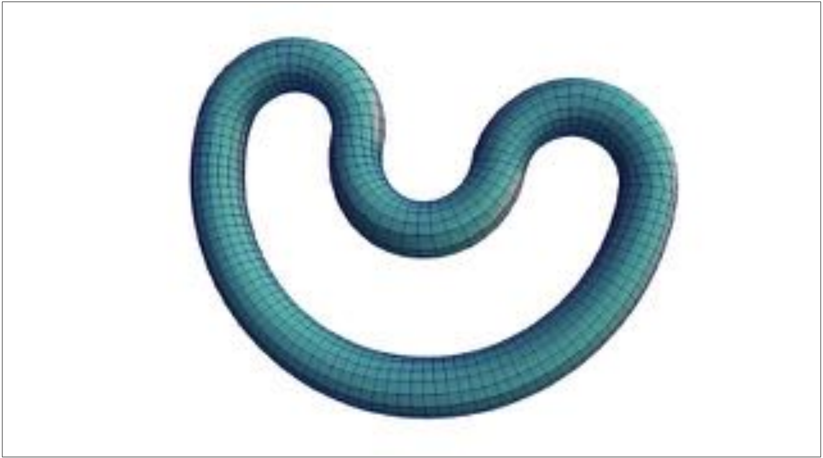




Notice that the surface fairing behavior is present still in this case, and the mesh remains nearly conformal throughout.

Results: Horseshoe

It is not necessary to restrict to genus 0 surfaces. Here is the Willmore flow of a horseshoe surface constrained by volume and surface area.



esh is corrected immediately by the flow.

Results: Knot

The Willmore evolution of a trefoil knot constrained by both surface area and enclosed volume.

Future work

Challenges:

- Find a reasonable way to conformally correct on the surface itself, not just its tangent space (higher-order approximation).
- Extend this notion of conformality to triangular meshes.
- Stabilize the p-Willmore flow for higher values of p.

Ideas to investigate:

- Build conformality directly into the flow equations.
- Compute a time-dependent holomorphic 1-form basis for T^*M and use it to conformally parametrize at each step.
- Extend these ideas to other curvature flows of interest (Ricci flow, Yamabe flow, etc.)

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